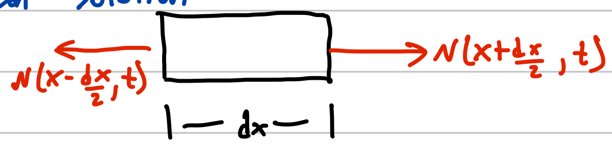
Analytical Solution



$\Sigma F = ma$

$$N(x+\tfrac{dx}{2},t) - N(x-\tfrac{dx}{2},t) = ma$$
$$= m\ddot{u}(x,t)$$
$$= \rho A dx\, \ddot{u}(x,t)$$

$$\frac{N(x+\frac{dx}{2},t) - N(x-\frac{dx}{2},t)}{dx} = \rho A \ddot{u}(x,t)$$

$$\frac{\partial N(x,t)}{\partial x} = \rho A \ddot{u}(x,t)$$

$$N = AE\frac{\partial u(x,t)}{\partial x}$$

$$\frac{\partial [AE\, \partial u/\partial x]}{\partial x} = \rho A \ddot{u}$$

$$AE\frac{\partial^2 u}{\partial x^2} = \rho A \frac{\partial^2 u}{\partial t^2}$$

assume $u(x,t) = U(x)\sin(\Omega t)$

$$AE[U''(x)\cancel{\sin(\Omega t)}] = \rho A[-U(x)\Omega^2 \cancel{\sin(\Omega t)}]$$

$$AEU''(x) = -\rho A \Omega^2 U(x)$$

$$\boxed{U''(x) + \frac{\rho\Omega^2}{E}U(x) = 0}$$

let $\alpha^2 = \frac{\rho\Omega^2}{E}$

$$\boxed{U''(x) + \alpha^2 U(x) = 0} \quad (1)$$

Specific Solution

$0 < x < 1 \quad U(0) = 0 \quad U(1) = 100$

dynamic equilibrium BCs

$$U''(x) + \alpha^2 U(x) = 0 \quad 0 < x < 1$$

$$\alpha^2 = \frac{\rho\Omega^2}{E}$$

$$U(x) = C_1\sin(\alpha x) + C_2\cos(\alpha x)$$

from $U(0) = 0$, $C_2 = 0$

from $U(1) = 100$, $C_1 = \frac{100}{\sin(\alpha)}$

$$\boxed{U(x) = \frac{100\sin(\alpha x)}{\sin(\alpha)}}$$

$$\boxed{U'(x) = \frac{100\alpha}{\sin(\alpha)}\cos(\alpha x)}$$

when $\alpha = \pm n\pi$, amplitude experiences resonance

$$\alpha = \Omega\sqrt{\frac{\rho}{E}} = \pm n\pi$$

$$\Omega = \pm n\pi\sqrt{\frac{E}{\rho}}$$

FDM $\quad U_{i-1} \quad U_i \quad U_{i+1}$

$i-1 \quad i \quad i+1$

From (1), $U''(x) = -\alpha^2 U(x)$

$U'''(x) = -\alpha^2 U'(x)$

$U''''(x) = -\alpha^2 U''(x) = \alpha^4 U(x)$

$$U_{i-1} = U_i - \Delta x U_i' + \frac{\Delta x^2 U_i''}{2} - \frac{\Delta x^3 U_i'''}{6} + \frac{\Delta x^4 U_i''''}{24}$$

$$U_{i+1} = U_i + \Delta x U_i' + \frac{\Delta x^2 U_i''}{2} + \frac{\Delta x^3 U_i'''}{6} + \frac{\Delta x^4 U_i''''}{24}$$

$+$

$$U_{i-1} + U_{i+1} = 2U_i + \Delta x^2 U_i'' + \frac{\Delta x^4 U_i''''}{12}$$

$$\Delta x^2 U_i'' = U_{i-1} - 2U_i + U_{i+1} - \frac{\Delta x^4 U_i''''}{12}$$

$$U_i'' = \underbrace{\frac{U_{i-1} - 2U_i + U_{i+1}}{\Delta x^2}}_{\text{2nd order}} \underbrace{- \frac{\Delta x^2}{12}\left(\alpha^4 U_i\right)}_{\text{4th order}}$$

2nd Order: $U_i'' = \left[\frac{1}{\Delta x^2}\right] U_{i-1} - \left[\frac{2}{\Delta x^2}\right] U_i + \left[\frac{1}{\Delta x^2}\right] U_{i+1}$ (2)

4th Order: $U_i'' = \left[\frac{1}{\Delta x^2}\right] U_{i-1} - \left[\frac{2}{\Delta x^2} + \frac{\Delta x^2 \alpha^4}{12}\right] U_i + \left[\frac{1}{\Delta x^2}\right] U_{i+1}$ (3)

$$\left.\begin{array}{l} U_i = \dfrac{100}{\sin(\bar{\alpha})} \sin(\bar{\alpha} x_i) \\ \ldots \cos(\bar{\alpha}\Delta x) = \dfrac{k}{2} \\ \bar{\alpha} = \dfrac{1}{\Delta x} \cos^{-1}\left(\dfrac{k}{2}\right) \\ -1 < \dfrac{k}{2} < 1 \end{array}\right]$$

to have a "correct" discrete solution

FDM

(2) into (1)

$$\left[\frac{1}{\Delta x^2}\right]U_{i-1} - \left[\frac{2}{\Delta x^2}\right]U_i + \left[\frac{1}{\Delta x^2}\right]U_{i+1} + \alpha^2 U_i = 0$$

$$\left[\frac{1}{\Delta x^2}\right]U_{i-1} - \left[\frac{2}{\Delta x^2} - \alpha^2\right]U_i + \left[\frac{1}{\Delta x^2}\right]U_{i+1} = 0$$

$$\boxed{U_{i-1} - \underbrace{\left[2 - \alpha^2\Delta x^2\right]}_{k_{fdm2} = 2 - \alpha^2\Delta x^2} U_i + U_{i+1} = 0}$$ (4) 2nd Order FDM

(3) into (1)

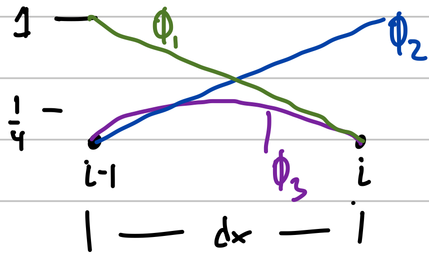
$$\left[\frac{1}{\Delta x^2}\right]U_{i-1} - \left[\frac{2}{\Delta x^2} + \frac{\Delta x^2\alpha^4}{12}\right]U_i + \left[\frac{1}{\Delta x^2}\right]U_{i+1} + \alpha^2 U_i = 0$$

$$\left[\frac{1}{\Delta x^2}\right]U_{i-1} - \left[\frac{2}{\Delta x^2} + \frac{\Delta x^2\alpha^4}{12} + \alpha^2\right]U_i + \left[\frac{1}{\Delta x^2}\right]U_{i+1} = 0$$

$$\boxed{U_{i-1} - \underbrace{\left[2 + \alpha^2\Delta x^2 + \frac{\Delta x^4\alpha^4}{12}\right]}_{k_{fdm4} = k_{fdm2} + \frac{\Delta x^4\alpha^4}{12}} U_i + U_{i+1} = 0}$$ (5) 4th Order FDM

FEM

$$\int_I (U'' + \alpha^2 U)\,dx = 0$$



$$\phi_1 = 1 - \frac{x}{\Delta x} \qquad \phi_2 = \frac{x}{\Delta x} \qquad \phi_3 = \phi_1 \cdot \phi_2 = \frac{x}{\Delta x} - \frac{x^2}{\Delta x^2}$$

$$\phi_1' = -\frac{1}{\Delta x} \qquad \phi_2' = \frac{1}{\Delta x} \qquad \phi_3' = \frac{1}{\Delta x} - \frac{2x}{\Delta x^2}$$

$$\int_I (U'' + \alpha^2 U)\phi\,dx = 0$$

$$\int_I U''\phi\,dx + \int_I \alpha^2 U \phi\,dx = 0$$

$$\int_I \left((U'\phi)' - U'\phi'\right)dx + \int_I \alpha^2 U\phi\,dx = 0$$

$$U'\phi\Big|_I - \int_I U'\phi'\,dx + \int_I \alpha^2 U\phi\,dx = 0$$

$$-\int_I U'\phi_m'\,dx + \int_I \alpha^2 U\phi_m\,dx = -U'\phi_m\Big|_I \quad \text{for} \quad \phi_m = \{\phi_1, \phi_2, \phi_3\}$$

$$U = U_{i-1}\phi_1 + U_i\phi_2 + \varepsilon\phi_3 \quad \text{some } \varepsilon \in \mathbb{R}$$

$$U' = U_{i-1}\phi_1' + U_i\phi_2' + \varepsilon\phi_3'$$

$$-\int_I \left(U_{i-1}\phi_1' + U_i\phi_2' + \mathcal{E}\phi_3'\right)\phi_m'\,dx + \int_I \alpha^2\left(U_{i-1}\phi_1 + U_i\phi_2 + \mathcal{E}\phi_3\right)\phi_m\,dx = -U'\phi_m\Big|_{i-1}^{i}$$

$$\left[\int_I \left(-\phi_1'\phi_m' + \alpha^2\phi_1\phi_m\right)dx\right]U_{i-1} + \left[\int_I \left(\phi_2'\phi_m' + \alpha^2\phi_2\phi_m\right)dx\right]U_i + \left[\int_I \left(-\phi_3'\phi_m' + \alpha^2\phi_3\phi_m\right)dx\right]\mathcal{E}_i = -U'\phi_m\Big|_{i-1}^{i}$$

m=1: $\left[\int_I \left(-\phi_1'^2 + \alpha^2\phi_1^2\right)dx\right]U_{i-1} + \left[\int_I \left(-\phi_1'\phi_2' + \alpha^2\phi_1\phi_2\right)dx\right]U_i + \left[\int_I \left(-\phi_1'\phi_3' + \alpha^2\phi_1\phi_3\right)dx\right]\mathcal{E}_i = -\left[0 - U_{i-1}'^{+}\right] = U_{i-1}'^{+}$

0

m=2: $\left[\int_I \left(-\phi_1'\phi_2' + \alpha^2\phi_1\phi_2\right)dx\right]U_{i-1} + \left[\int_I \left(-\phi_2'^2 + \alpha^2\phi_2^2\right)dx\right]U_i + \left[\int_I \left(-\phi_2'\phi_3' + \alpha^2\phi_2\phi_3\right)dx\right]\mathcal{E}_i = -\left[U_i'^{-} - 0\right] = -U_i'^{-}$

m=3: $\left[\int_I \left(-\phi_1'\phi_3' + \alpha^2\phi_1\phi_3\right)dx\right]U_{i-1} + \left[\int_I \left(-\phi_2'\phi_3' + \alpha^2\phi_2\phi_3\right)dx\right]U_i + \left[\int_I \left(-\phi_3'^2 + \alpha^2\phi_3^2\right)dx\right]\mathcal{E}_i = 0$

$$\begin{bmatrix} k_{11}^i & k_{12}^i & k_{13}^i \\ k_{21}^i & k_{22}^i & l_{23}^i \\ k_{31}^i & k_{32}^i & k_{33}^i \end{bmatrix}\begin{bmatrix} U_{i-1} \\ U_i \\ \mathcal{E}_i \end{bmatrix} = \begin{bmatrix} U_{i-1}'^{+} \\ -U_i'^{-} \\ 0 \end{bmatrix}$$

$N_i^- = N_i^+ \quad \therefore (AE)_i U_i'^{-} - (AE)_{i+1} U_i'^{+} = 0$

$$\begin{bmatrix} k_{11}^{i+1} & k_{12}^{i+1} & k_{13}^{i+1} \\ k_{21}^{i+1} & k_{22}^{i+1} & k_{23}^{i+1} \\ k_{31}^{i+1} & k_{32}^{i+1} & k_{33}^{i+1} \end{bmatrix}\begin{bmatrix} U_i \\ U_{i+1} \\ \mathcal{E}_{i+1} \end{bmatrix} \quad \begin{bmatrix} U_i'^{-} \\ -U_{i+1}'^{+} \\ 0 \end{bmatrix}$$

$(AE)_i = (AE)_{i+1}$ for prismatic bar of 1 material

$$\left.\begin{aligned} &\boxed{k_{21}^i U_{i-1} + \left(k_{22}^i + k_{12}^{i+1}\right)U_i + \left(k_{23}^{i+1}\right)U_{i+1} + k_{13}^i\mathcal{E}_i + k_{23}^{i+1}\mathcal{E}_{i+1} = 0} \\ &\boxed{k_{31}^i U_{i-1} + k_{32}^i U_i + k_{33}^i \mathcal{E}_i = 0} \end{aligned}\right\} P = 2 \text{ FEM}$$

$$\boxed{k^{i}_{21}\, U_{i-1} + \left(k^{i}_{22} + k^{i+1}_{11}\right) U_i + \left(k^{i+1}_{12}\right) U_{i+1} = 0}$$

P=1 FEM

$$k_{mn} = \int_I \left(-\phi_m' \phi_n' + \alpha^2 \phi_m \phi_n\right) dx$$

$$\phi_1 = 1 - \frac{x}{\Delta x} \qquad \phi_2 = \frac{x}{\Delta x} \qquad \phi_3 = \phi_1 \cdot \phi_2 = \frac{x}{\Delta x} - \frac{x^2}{\Delta x^2}$$

$$\phi_1' = -\frac{1}{\Delta x} \qquad \phi_2' = \frac{1}{\Delta x} \qquad \phi_3' = \frac{1}{\Delta x} - \frac{2x}{\Delta x^2}$$

$$\boxed{\begin{array}{ll}
k_{11} = \dfrac{-1}{\Delta x} + \dfrac{\alpha^2 \Delta x}{3} & k_{12} = k_{21} = \dfrac{1}{\Delta x} + \dfrac{\alpha^2 \Delta x}{6} \\[2ex]
k_{22} = \dfrac{-1}{\Delta x} + \dfrac{\alpha^2 \Delta x}{3} & k_{13} = k_{31} = 0 + \dfrac{\alpha^2 \Delta x}{12} \\[2ex]
k_{33} = \dfrac{-1}{3\Delta x} + \dfrac{\alpha^2 \Delta x}{30} & k_{23} = k_{32} = 0 + \dfrac{\alpha^2 \Delta x}{12}
\end{array}}$$

4th Order U_i' for Nodes Only

$$U_{i-1} = U_i - \Delta x U_i' + \frac{\Delta x U_i''}{2} - \frac{\Delta x^3 U_i'''}{6} + \frac{\Delta x}{24} U_i''''$$

$$\Delta x U_i' = U_i - U_{i-1} + \frac{\Delta x^2}{2}\left(-\alpha^2 U_i\right) - \frac{\Delta x^3}{6}\left(-\alpha^2 U_i'\right) + \frac{\Delta x^4}{24}\left(\alpha^4 U_i\right)$$

$$\left[\Delta x - \frac{\Delta x^3 \alpha^2}{6}\right] U_i' = \left[1 - \frac{\Delta x^2 \alpha^2}{2} + \frac{\Delta x^4 \alpha^4}{24}\right] U_i - U_{i-1}$$

$$\boxed{U_i' = \frac{\left[1 - \frac{\Delta x^2 \alpha^2}{2} + \frac{\Delta x^4 \alpha^4}{24}\right]}{\left[\Delta x - \frac{\Delta x^3 \alpha^2}{6}\right]} U_i - \frac{1}{\left[\Delta x - \frac{\Delta x^3 \alpha^2}{6}\right]} U_{i-1}} \quad (6)$$

$$U_{i+1} = U_i + \Delta x U_i' + \frac{\Delta x^2 U_i''}{2} + \frac{\Delta x^3 U_i'''}{6} + \frac{\Delta x^4 U_i''''}{24}$$

$$U_{i+1} = U_i + \Delta x U_i' - \frac{\Delta x^2 \alpha^2 U_i}{2} - \frac{\Delta x^3 \alpha^2 U_i'}{6} + \frac{\Delta x^4 \alpha^4 U_i}{24}$$

$$\left[-\Delta x + \frac{\Delta x^3 \alpha^2}{6}\right] U_i' = \left[1 - \frac{\Delta x^2 \alpha^2}{2} + \frac{\Delta x^4 \alpha^2}{24}\right] U_i - U_{i+1}$$

$$\boxed{U_i' = \frac{\left[1 - \frac{\Delta x^2 \alpha^2}{2} + \frac{\Delta x^4 \alpha^2}{24}\right]}{\left[-\Delta x + \frac{\Delta x^3 \alpha^2}{6}\right]} U_i - \frac{1}{\left[-\Delta x + \frac{\Delta x^3 \alpha^2}{6}\right]} U_{i+1}} \quad (7)$$

4th Order $U(x)$ Interpolation

$$U(x_i + h) = U_i + hU_i' + \frac{h^2 U_i''}{2} + \frac{h^3 U_i'''}{6} + \frac{h^4 U_i''''}{24}$$

$$U(x_i + h) = U_i + hU_i' - \frac{h^2 \alpha^2}{2} U_i - \frac{h^3 \alpha^2 U_i}{6} + \frac{h^4 \alpha^4 U_i}{24}$$

$$\boxed{U(x_i + h) = \left[1 - \frac{h^2 \alpha^2}{2} + \frac{h^4 \alpha^4}{24}\right] U_i + \left[h - \frac{h^3 \alpha^2}{6}\right] U_i'} \quad (6) \text{ or } (7)$$

4th Order $U'(x)$

$$U'(x_i+h) = U_i' + hU_i'' + \frac{h^2}{2}U_i''' + \frac{h^3}{6}U_i'''' + \frac{h^4}{24}U_i'''''$$

$$U'(x_i+h) = U_i' - h\alpha^2 U_i - \frac{h^2}{2}\alpha^2 U_i' + \frac{h^3\alpha^4}{6}U_i + \frac{h^4\alpha^4 U_i'}{24}$$

$$\boxed{U'(x_i+h) = \left[\frac{h^3\alpha^4}{6} - h\alpha^2\right]U_i + \left[1 - \frac{h^2}{2}\alpha^2 + \frac{h^4\alpha^4}{24}\right]U_i'}$$ ↙ (6) or (7)

Simpler Constitutive Relation

$$N_i^- = (AE)_1 U_i'$$

$$(AE)_1 U_i' - (AE)_2 U_i^{+} = 0$$