AERO 430 HW 4

Judson Upchurch

2024-11-06

Table of Contents

1 Analytical Solution

1.1 Abstract

This section provides an analytical solution for the temperature distribution of a bimaterial beam connected at an interface point. The beam is constrained such that the environment is at 0 degrees, the right end of the beam is fixed at 100 degrees, and the left is fixed at 0 degrees.
In this report, the beam consists of two sections with the same area but different thermal conductivities. The affects of varying the ratio between the thermal conductivities will be analyzed as well as changing the interface location.

1.2 Derivation of General Solution

Let us first look at derivation for the general solution for temperature as a function along the rod. To begin, the governing equation will be conservation of heat flux.

1.2.1 Conservation of Heat Flux

In the case of a rod without internal heat generation, conservation of heat flux reduces to the total heat flux entering the system is equal and opposite to the heat flux leaving the system.

$$\dot{Q}_{out}(x) - \dot{Q}_{in}(x) = 0 \tag{1}$$

$\dot{Q}$ is the rate of heat transfer in energy / second units. We will use Joules/second, aka Watts.

1.2.2 Newton's Law of Cooling

Newton's Law of Cooling shows that the rate of heat transfer to the surroundings is proportional to the area of heat transfer and the temperature difference between the object and the surroundings.

$$\dot{Q}_{env}(x) = hA\left(T(x) - T_{amb}\right) \tag{2}$$

$\dot{Q}_{env}(x)$ is the rate of heat flux to the environment at a point along the rod in Joules/second. h is the heat transfer coefficient of the environment's fluid in $\frac{W}{m^2\,^\circ C}$. A is the area exposed to the surroundings in m^2. $T(x)$ is the temperature of the rod at position x in $^\circ C$.

In both of our cases, the ambient temperature, T_{amb} is 0 $^\circ C$. This reduces the equation for Newton's Law of Cooling to:

$$\dot{Q}_{env}(x) = hAT(x) \tag{3}$$

1.2.3 Fourier's Law of Conduction

Fourier's Law of Conduction states that the rate of heat transfer through a material via conduction is proportional to the area of heat transfer and the negative temperature gradient.

$$\dot{Q}_c(x) = -kA\frac{dT(x)}{dx} \tag{4}$$

$\dot{Q}_c(x)$ is the rate of heat transfer via conduction through the material in Joules/second. k is the thermal conductivity of the rod in $\frac{W}{m^\circ C}$. A is the cross-sectional area of the rod in m^2. $\frac{dT(x)}{dx}$ is the temperature gradient of the rod in $\frac{^\circ C}{m}$.

1.2.4 General Solution

If we look at an infinitesimal slice of the rod starting at position x and total length dx, we use conservation of heat flux (Equation 1) to get:

$$\dot{Q}(x+dx) - \dot{Q}(x) + \dot{Q}_{lat}(x;dx) = 0$$

From left to right, we have the heat flux leaving the right side of the rod slice, the heat flux entering the left side of the rod slice, and the heat flux leaving the rod slice from the outer circumference to the environment. Replacing $\dot{Q}_{lat}(x;dx)$ with $hPdxT(x)$ from Newton's Law of Cooling (Equation 3) gives:

$$\dot{Q}(x+dx) - \dot{Q}(x) + h(Pdx)T(x) = 0$$

Pdx is the area exposed to the surroundings for a given length dx where P is the perimeter. Dividing the equation by dx gives us:

$$\frac{\dot{Q}(x+dx) - \dot{Q}(x)}{dx} + hPT(x) = 0$$

Noting that $\frac{\dot{Q}(x+dx)-\dot{Q}(x)}{dx} = \dot{Q}'(x)$ as $dx \to 0$, the equation becomes:

$$\dot{Q}'(x) + hPT(x) = 0$$

Replacing $\dot{Q}(x)$ with $-kAT'(x)$ from Equation 4 gives:

$$-kAT''(x) + hPT(x) = 0$$

Rearranging and letting $\alpha = \sqrt{\frac{hP}{kA}}$ gives:

$$T''(x) - \alpha^2 T(x) = 0 \tag{5}$$

This will be the governing differential equation for both cases. Solving Equation 5 using the characteristic equation $r^2 - \alpha^2 = 0$ gives:

$$T(x) = Ae^{\alpha x} + Be^{-\alpha x}$$

Using the definitions $\sinh(\omega x) = \frac{e^{\omega x}-e^{-\omega x}}{2}$ and $\cosh(\omega x) = \frac{e^{\omega x}+e^{-\omega x}}{2}$, we get:

$$T(x) = C\sinh(\alpha x) + D\cosh(\alpha x) \tag{6}$$

Due to the interface in the bar, denoted at $\bar{x}$, the general solution is applied twice over each section with independent α_1 and α_2.

$$\begin{aligned} T_1(x) &= C_1 \sinh(\alpha_1 x) + D_1 \cosh(\alpha_1 x) \quad \text{for} \quad 0 \leq x \leq \bar{x} \\ T_2(x) &= C_2 \sinh(\alpha_2 x) + D_2 \cosh(\alpha_2 x) \quad \text{for} \quad \bar{x} \leq x \leq L \end{aligned} \tag{7}$$

The constants C_1, D_1, C_2, and D_2 must be solved simultaneously using the conditions that $T_1(0) = 0$, $T_1(L) = 100$, $T_1(\bar{x}) = T_2(\bar{x})$, and $-k_1 A_1 T_1'(\bar{x}) = k_2 A_2 T_2'(\bar{x})$.

1.3 Interface Temperature Results

1.3.1 Interface Location 1: $\bar{x} = 0.5$

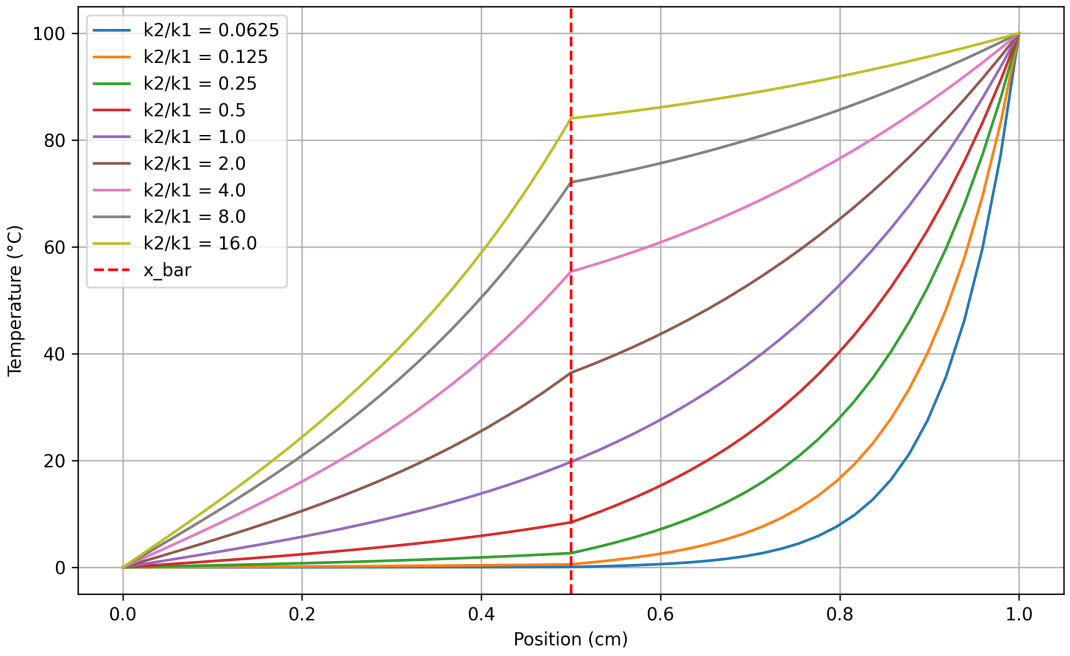

Figure 1: Analytical Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

Table 1: Analytical Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

x	$\frac{k_2}{k_1}$ 0.0625	0.125	0.25	0.5	1.0	2.0	4.0	8.0	16.0
0.000	0	0	0	0	0	0	0	0	0
0.125	0.0117	0.0976	0.4648	1.471	3.440	6.352	9.653	12.57	14.65
0.250	0.0252	0.2107	1.003	3.175	7.425	13.71	20.83	27.13	31.63
0.375	0.0427	0.3572	1.700	5.381	12.59	23.24	35.31	45.98	53.60
0.500	0.0669	0.5602	2.667	8.439	19.74	36.44	55.38	72.11	84.07
0.625	0.8478	3.304	8.627	17.44	30.02	45.82	62.46	76.70	86.74
0.750	4.228	10.63	20.26	32.03	45.04	58.79	71.99	82.80	90.26
0.875	20.57	32.67	45.24	56.89	67.20	76.39	84.34	90.51	94.67
1.000	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

1.3.2 Interface Location 2: $\bar{x} = \frac{2}{pi}$

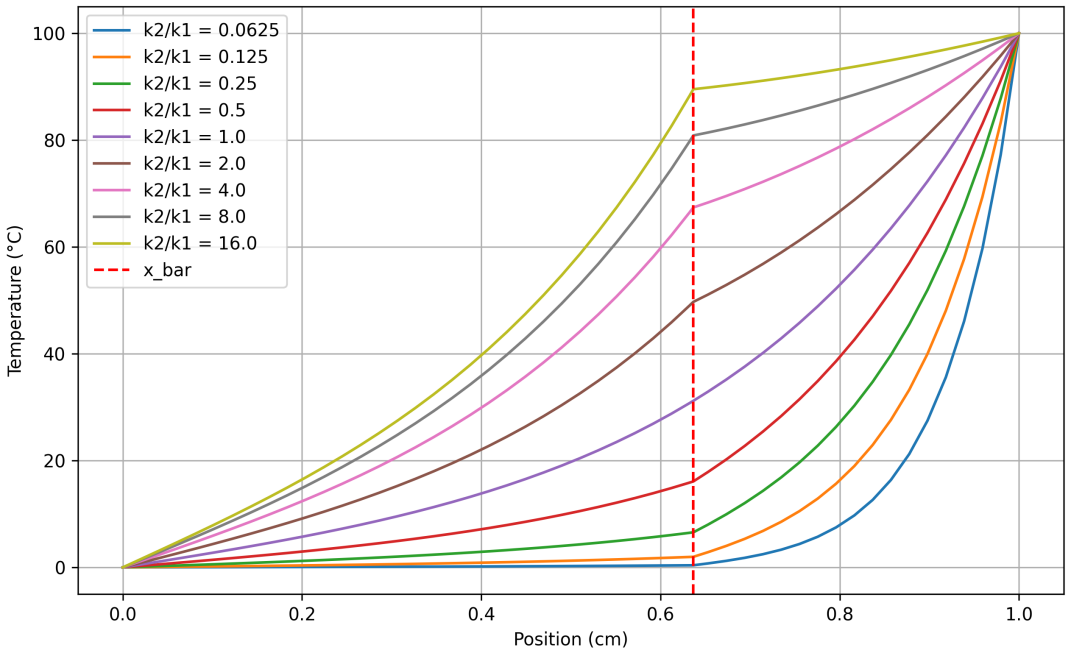

Figure 2: Analytical Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

Table 2: Analytical Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

x	$\frac{k_2}{k_1}$ 0.0625	0.125	0.25	0.5	1.0	2.0	4.0	8.0	16.0
0.000	0	0	0	0	0	0	0	0	0
0.125	0.0433	0.2178	0.7238	1.775	3.440	5.484	7.435	8.922	9.877
0.250	0.0934	0.4700	1.562	3.831	7.425	11.84	16.05	19.26	21.32
0.375	0.158	0.7967	2.648	6.494	12.59	20.06	27.20	32.64	36.13
0.500	0.248	1.249	4.153	10.18	19.74	31.46	42.65	51.19	56.67
0.625	0.377	1.900	6.315	15.49	30.02	47.85	64.86	77.84	86.18
0.637	0.392	1.974	6.561	16.09	31.18	49.71	67.38	80.86	89.53
0.750	4.086	10.00	18.93	30.68	45.04	60.65	74.76	85.28	91.97
0.875	20.55	32.49	44.74	56.31	67.20	77.28	85.70	91.74	95.52
1.000	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

1.3.3 Discussion

It can be seen from Figure 1 and Figure 2 that the section with lower thermal conductivity see a more gradual change in temperature compared to a higher k value. Additionally, comparing the two solutions with different interface locations shows a larger instantaneous temperature gradient at the interface.

1.4 Heat Convection Results

1.4.1 Interface Location 1: $\bar{x} = 0.5$

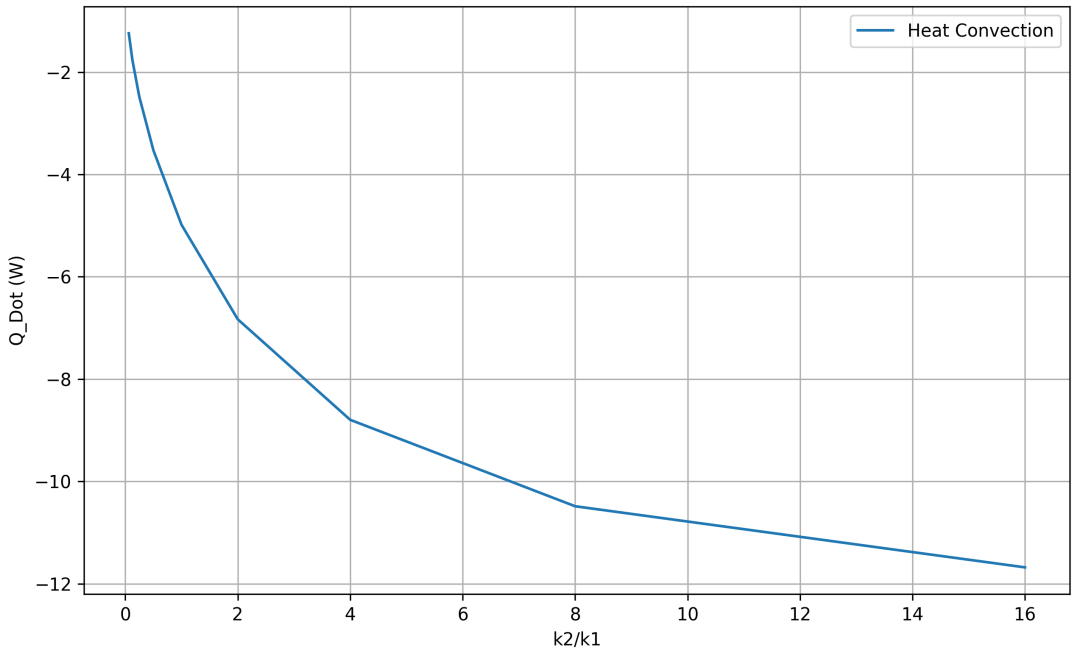

Figure 3: Analytical Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

Table 3: Analytical Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

$\frac{k_2}{k_1}$	Heat Convection
0.0625	-1.242
0.125	-1.756
0.25	-2.487
0.5	-3.529
1.0	-4.985
2.0	-6.832
4.0	-8.797
8.0	-10.49
16.0	-11.68

1.4.2 Interface Location 2: $\bar{x} = \frac{2}{pi}$

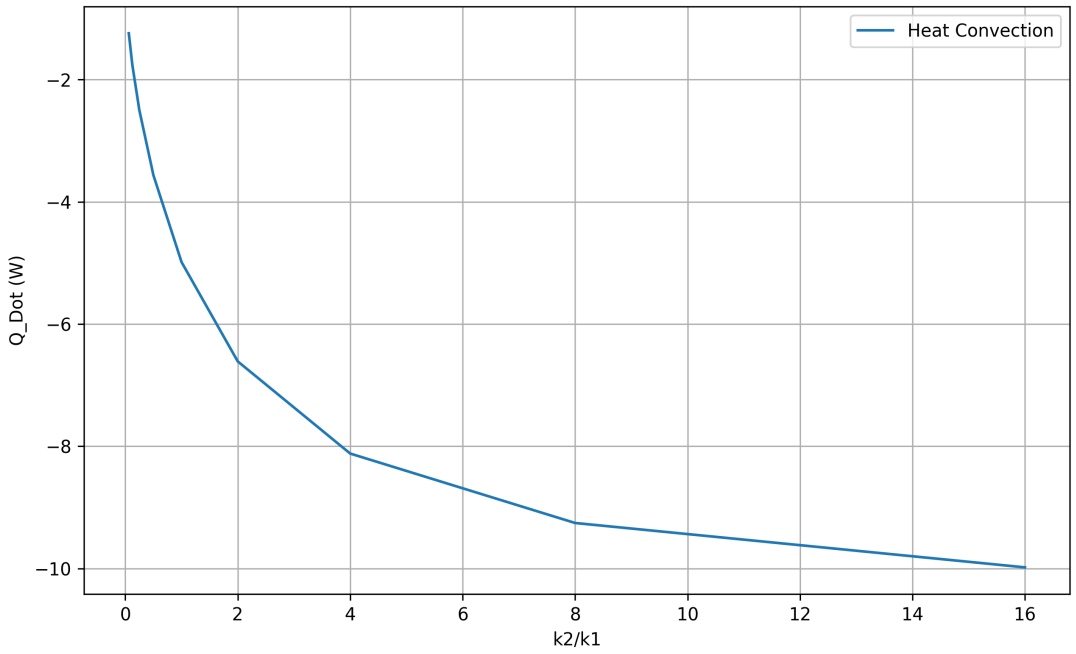

Figure 4: Analytical Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

Table 4: Analytical Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

$\frac{k_2}{k_1}$	Heat Convection
0.0625	-1.242
0.125	-1.759
0.25	-2.501
0.5	-3.564
1.0	-4.985
2.0	-6.611
4.0	-8.119
8.0	-9.255
16.0	-9.981

1.4.3 Discussion

From Figure 3 and Figure 4, there is larger heat loss due to convection when the thermal conductivity of the right section of the beam becomes larger.

2 FDM Solution

2.1 Abstract

The finite difference method proves useful in solving the temperature distribution problem of a heat conducting bar. Due to the simplicity of the governing differential equation, the Taylor Expansion can be computed with arbitrarily high terms.

This section of the report will deal with the 2nd order and 4th order finite difference method for obtaining the temperature distribution and heat convection of the end of the bar.

2.2 FDM Derivation

Given the governing ODE:

$$T'' - \alpha^2 T = 0$$

where $\alpha^2 = \frac{hp}{kA}$, we will derive finite difference approximations for both second- and fourth-order accuracy at a node i with non-uniform spacing.
Consider a node i with adjacent nodes $i-1$ and $i+1$. On the left side, we have properties k_1, A_1, and spacing Δx_1; on the right side, we have k_2, A_2, and spacing Δx_2.
To approximate the heat flux at node i, we start with Taylor expansions.
For the left side:

$$T_{i-1} = T_i - \Delta x_1 T_i' + \frac{\Delta x_1^2}{2} T_i'' - \frac{\Delta x_1^3}{6} T_i''' + \frac{\Delta x_1^4}{24} T_i^{(4)} + \mathcal{O}(\Delta x_1^5)$$

Solving for T'^- (left-side derivative):

$$T'^- \approx \frac{T_i - T_{i-1}}{\Delta x_1} - \frac{\Delta x_1}{2} T_i'' + \frac{\Delta x_1^2}{6} T_i''' - \frac{\Delta x_1^3}{24} T_i^{(4)}$$

For the right side:

$$T_{i+1} = T_i + \Delta x_2 T_i' + \frac{\Delta x_2^2}{2} T_i'' + \frac{\Delta x_2^3}{6} T_i''' + \frac{\Delta x_2^4}{24} T_i^{(4)} + \mathcal{O}(\Delta x_2^5)$$

Solving for T'^{+} (right-side derivative):

$$T'^{+} \approx \frac{T_{i+1} - T_i}{\Delta x_2} - \frac{\Delta x_2}{2} T_i'' + \frac{\Delta x_2^2}{6} T_i''' - \frac{\Delta x_2^3}{24} T_i^{(4)}$$

Using the governing ODE $T'' = \alpha^2 T$ and its derivatives $T''' = \alpha^2 T'$ and $T^{(4)} = \alpha^4 T$, we substitute into the expressions for T'^{-} and T'^{+}.
For the left-side derivative (T'^{-}):

$$T'^{-} \approx \frac{T_i - T_{i-1}}{\Delta x_1} - \frac{\Delta x_1}{2} \alpha^2 T_i + \frac{\Delta x_1^2}{6} \alpha^2 T_i' - \frac{\Delta x_1^3}{24} \alpha^4 T_i$$

For the right-side derivative (T'^{+}):

$$T'^{+} \approx \frac{T_{i+1} - T_i}{\Delta x_2} - \frac{\Delta x_2}{2} \alpha^2 T_i + \frac{\Delta x_2^2}{6} \alpha^2 T_i' - \frac{\Delta x_2^3}{24} \alpha^4 T_i$$

For second-order accuracy, we neglect higher-order terms.
According to the conservation of heat flux (Equation 1):

$$k_1 A_1 T'^{-} - k_2 A_2 T'^{+} = 0$$

Substituting the expressions for T'^{-} and T'^{+}, we expand and rearrange terms to get:

$$a T_{i-1} + b T_i + c T_{i+1} = 0 \tag{8}$$

where the coefficients a, b, and c are defined as follows:
For second-order accuracy (neglecting higher-order terms involving α^2):

$$\begin{aligned} a &= -\frac{k_1 A_1}{\Delta x_1} \\ b &= \frac{k_1 A_1}{\Delta x_1} + \frac{k_2 A_2}{\Delta x_2} \\ c &= -\frac{k_2 A_2}{\Delta x_2} \end{aligned} \tag{9}$$

Thus, the second-order finite difference equation is:

$$a T_{i-1} + b T_i + c T_{i+1} = 0$$

For fourth-order accuracy (including all terms):

$$\begin{aligned} a &= -\frac{k_1 A_1}{\Delta x_1} \\ b &= \frac{k_1 A_1}{\Delta x_1} + \frac{k_2 A_2}{\Delta x_2} - \frac{k_1 A_1 \Delta x_1}{2} \alpha^2 - \frac{k_2 A_2 \Delta x_2}{2} \alpha^2 \\ c &= -\frac{k_2 A_2}{\Delta x_2} \end{aligned} \tag{10}$$

The fourth-order finite difference equation is:

$$a T_{i-1} + b T_i + c T_{i+1} = 0$$

For an example with N nodes, we consider N temperatures $T_1, T_2, \ldots, T_{N-2}$ as unknowns, and we apply boundary conditions $T_0 = T_0$ and $T_{N-1} = T_{N-1}$.

The finite difference equations can be arranged into a system of equations:

$$\begin{bmatrix} b_1 & c_2 & 0 & \cdots & 0 & 0 \\ a_2 & b_2 & c_3 & \cdots & 0 & 0 \\ 0 & a_3 & b_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{N-3} & c_{N-2} \\ 0 & 0 & 0 & \cdots & a_{N-2} & b_{N-2} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_{N-3} \\ T_{N-2} \end{bmatrix} = \begin{bmatrix} -a_1 T_0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -c_{N-1} T_{N-1} \end{bmatrix}$$

where the boundary conditions T_0 and T_{N-1} appear in the first and last equations, making them known values in the solution vector.

In this matrix, each row corresponds to a finite difference equation at a node i, with a_i, b_i, and c_i applied based on either second- or fourth-order accuracy, as defined earlier.

2.3 Interface Temperature Results

2.3.1 Interface Location 1: $\bar{x} = 0.5$

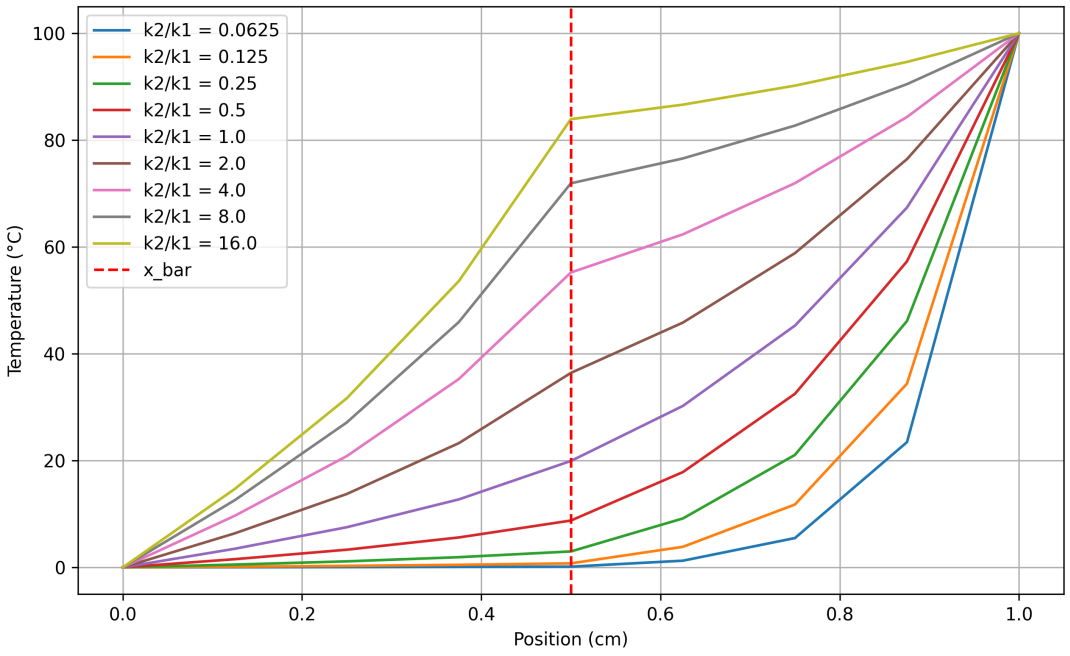

Figure 5: 2nd Order FDM Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

Table 5: 2nd Order FDM Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

x	$\frac{k_2}{k_1}$ 0.0625	0.125	0.25	0.5	1.0	2.0	4.0	8.0	16.0
0.000	0	0	0	0	0	0	0	0	0
0.125	0.0236	0.1312	0.5222	1.535	3.487	6.374	9.664	12.59	14.69
0.250	0.0508	0.2828	1.126	3.309	7.520	13.74	20.84	27.14	31.68
0.375	0.0860	0.4787	1.906	5.601	12.73	23.26	35.27	45.94	53.61
0.500	0.1346	0.7493	2.983	8.768	19.92	36.41	55.21	71.91	83.92
0.625	1.249	3.851	9.158	17.84	30.23	45.83	62.35	76.56	86.63
0.750	5.487	11.77	21.06	32.49	45.26	58.83	71.93	82.71	90.19
0.875	23.44	34.39	46.12	57.29	67.37	76.43	84.32	90.47	94.64
1.000	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

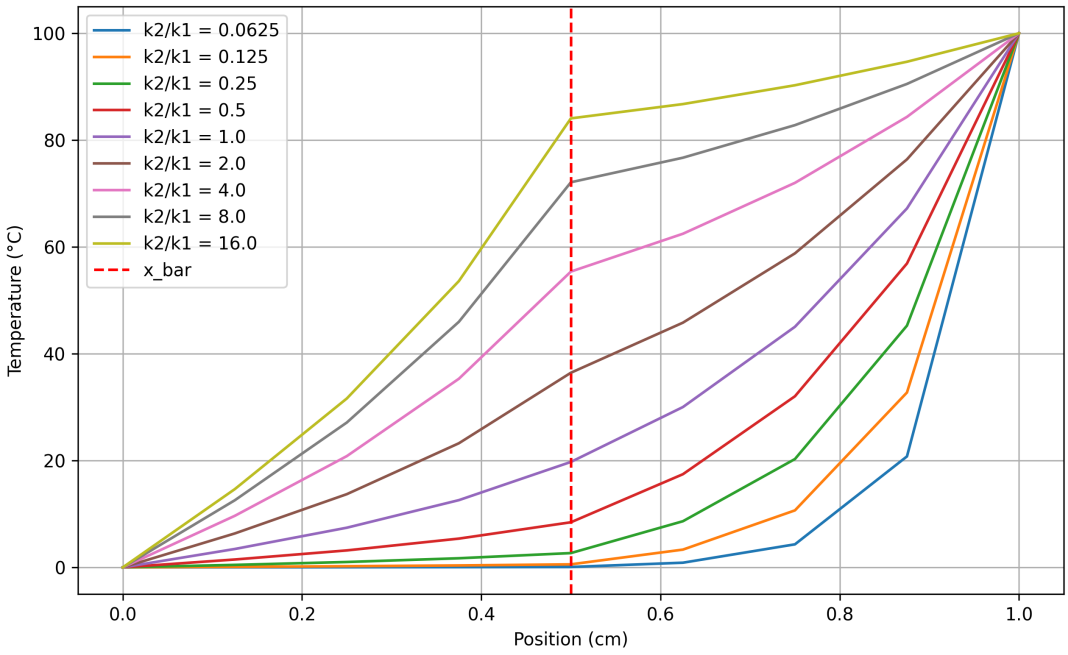

Figure 6: 4th Order FDM Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

Table 6: 4th Order FDM Temperature Distribution for Varying $\frac{k_2}{k_1}$ with $\bar{x} = 0.5$

x	$\frac{k_2}{k_1}$ 0.0625	0.125	0.25	0.5	1.0	2.0	4.0	8.0	16.0
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.125	0.01241	0.09906	0.4662	1.472	3.441	6.352	9.653	12.57	14.65
0.250	0.02678	0.2138	1.006	3.176	7.426	13.71	20.83	27.13	31.63
0.375	0.04539	0.3624	1.706	5.384	12.59	23.24	35.31	45.98	53.60
0.500	0.07118	0.5683	2.675	8.443	19.74	36.44	55.38	72.10	84.07
0.625	0.8730	3.325	8.638	17.44	30.02	45.81	62.46	76.70	86.74
0.750	4.312	10.67	20.28	32.03	45.05	58.79	71.99	82.80	90.26
0.875	20.78	32.74	45.26	56.90	67.20	76.39	84.34	90.51	94.67
1.000	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

2.3.2 Interface Location 2: $\bar{x} = \frac{2}{pi}$

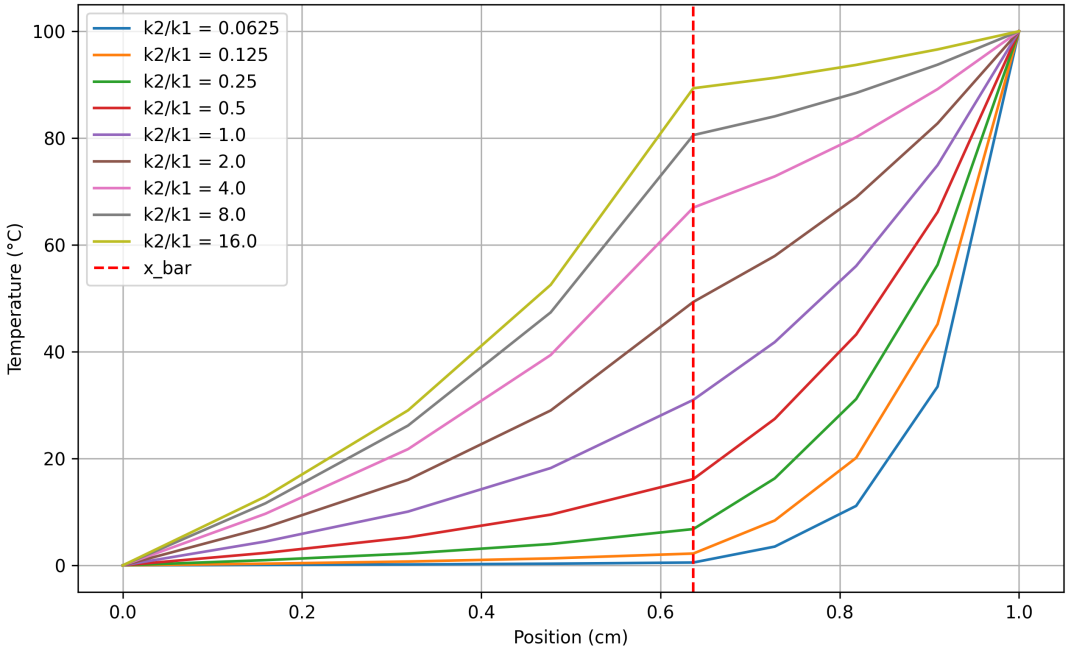

Figure 7: 2nd Order FDM Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

Table 7: 2nd Order FDM Temperature Distribution for Varying $\frac{k_2}{k_1}$ with $\bar{x} = \frac{2}{\pi}$

x	$\frac{k_2}{k_1}$ 0.0625	0.125	0.25	0.5	1.0	2.0	4.0	8.0	16.0
0.000	0	0	0	0	0	0	0	0	0
0.159	0.0771	0.3197	0.9805	2.330	4.469	7.113	9.660	11.62	12.88
0.318	0.1736	0.7204	2.209	5.250	10.07	16.03	21.77	26.18	29.03
0.477	0.3142	1.304	3.998	9.501	18.22	29.00	39.39	47.37	52.53
0.637	0.5343	2.217	6.799	16.16	30.99	49.33	66.99	80.56	89.33
0.727	3.516	8.402	16.28	27.43	41.80	57.93	72.83	84.07	91.28
0.818	11.14	20.13	31.14	43.22	56.05	68.92	80.17	88.45	93.70
0.909	33.47	45.16	56.28	66.15	74.93	82.75	89.16	93.74	96.60
1.000	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

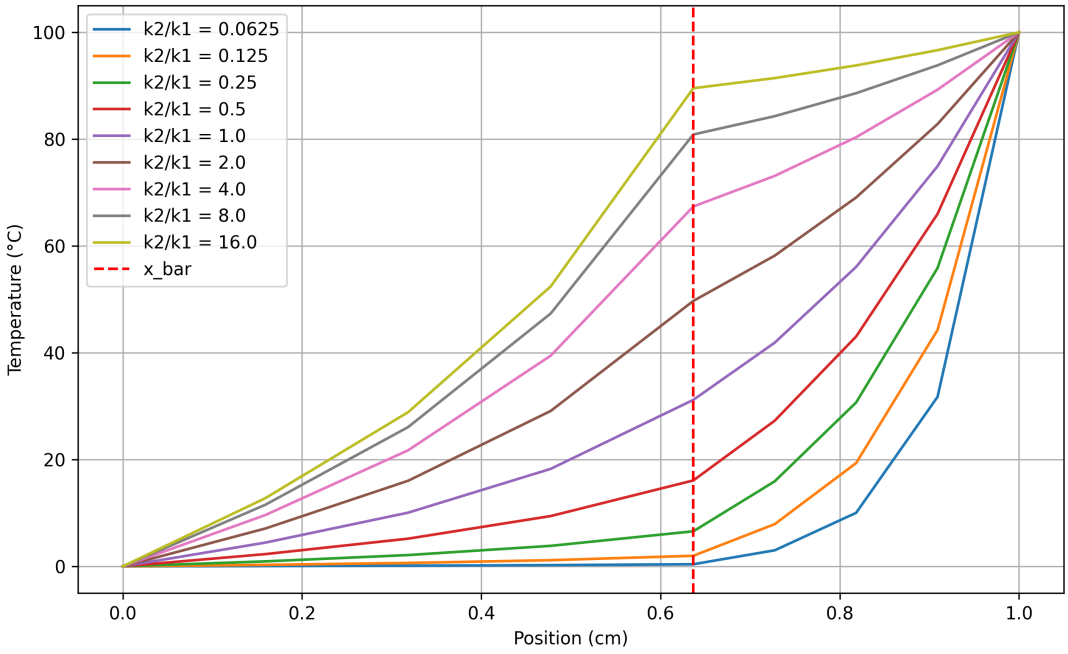

Figure 8: 4th Order Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

Table 8: 4th order FDM Temperature Distribution for Varying $\frac{k_2}{k_1}$ with $\bar{x} = \frac{2}{\pi}$

x	$\frac{k_2}{k_1}$ 0.0625	0.125	0.25	0.5	1.0	2.0	4.0	8.0	16.0
0.000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.159	0.05691	0.2827	0.9370	2.297	4.451	7.095	9.618	11.54	12.78
0.318	0.1285	0.6386	2.116	5.187	10.05	16.02	21.72	26.07	28.86
0.477	0.2334	1.160	3.843	9.419	18.25	29.10	39.45	47.34	52.41
0.637	0.3986	1.981	6.564	16.09	31.18	49.70	67.38	80.86	89.52
0.727	3.009	7.909	15.93	27.28	41.89	58.18	73.11	84.29	91.42
0.818	10.03	19.35	30.69	43.04	56.08	69.07	80.35	88.60	93.79
0.909	31.75	44.26	55.87	66.00	74.92	82.82	89.25	93.81	96.65
1.000	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

2.3.3 Discussion

From the results in **??**, Figure 6, Figure 7, and **??**, both FEM results for both cases showed similar results to the analytical temperature distribution in Figure 1 and Figure 2.

2.4 Heat Convection Results

2.4.1 Interface Location 1: $\bar{x} = 0.5$

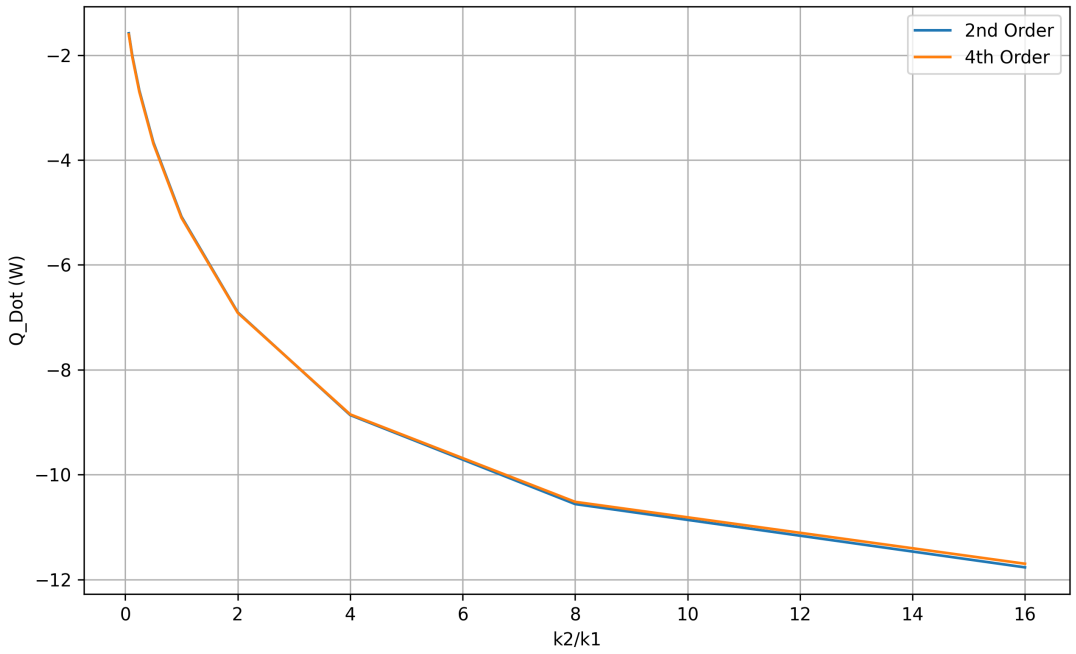

Figure 9: 2nd and 4th Order FDM Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

Table 9: 2nd and 4th Order FDM Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

$\frac{k_2}{k_1}$	2nd Order FDM Q	4th Order FDM Q
0.0625	-1.58304	-1.60397
0.125	-2.01235	-2.03826
0.25	-2.67453	-2.70145
0.5	-3.66509	-3.69011
1.0	-5.08231	-5.10301
2.0	-6.90522	-6.91495
4.0	-8.86402	-8.85149
8.0	-10.5613	-10.5193
16.0	-11.7687	-11.6995

2.4.2 Interface Location 2: $\bar{x} = \frac{2}{pi}$

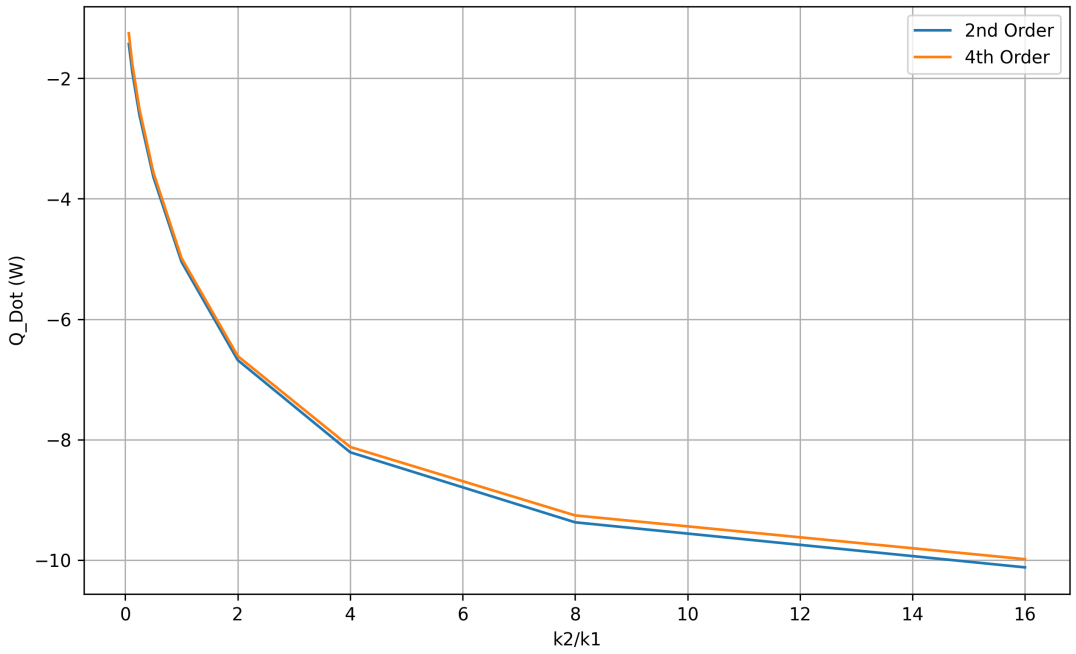

Figure 10: 2nd and 4th Order FDM Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

Table 10: 2nd and 4th Order FDM Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

$\frac{k_2}{k_1}$	2nd Order FDM Q	4th Order FDM Q
0.0625	-1.43246	-1.25370
0.125	-1.89880	-1.76348
0.25	-2.60334	-2.50298
0.5	-3.63985	-3.56491
1.0	-5.04765	-4.98555
2.0	-6.67856	-6.61142
4.0	-8.20809	-8.11975
8.0	-9.37058	-9.25606
16.0	-10.1183	-9.98244

2.4.3 Discussion

Looking at the heat convection results in Figure 9 and Figure 10 compared to Figure 3 and Figure 4 show similar heat convection results at the end of the bar for both methods and interface locations.

3 FEM Solution

3.1 Abstract

The finite element method adds an additional finite method for obtaining the temperature distribution of a heat conducting bar. By changing the basis function applied at each finite element, higher rates of convergence can be achieved.

This section of the report will deal with the P=1 and P=2 finite element methods for obtaining the temperature distribution and heat convection of the end of the bar.

3.2 FEM Derivation

The governing equation for a heat-conducting bar is:

$$\dot{Q}' + hPT = 0 \tag{11}$$

where $\dot{Q}'$ represents the rate of change of heat flux, h is the heat transfer coefficient, P is the perimeter of the bar, and T is the temperature.
To apply the weak formulation, we integrate over an element I:

$$\int_I \left(\dot{Q}' + hPT\right) v\,dx = 0 \tag{12}$$

where v is the test function.
Since $\dot{Q}' = (kAT'')$, where k is the thermal conductivity, A is the cross-sectional area, and T'' is the second derivative of temperature with respect to x, we substitute:

$$\int_I \left(kAT'' + hPT\right) v\,dx = 0 \tag{13}$$

To eliminate the second derivative T'', we perform integration by parts. Integrate $kAT''v$ by parts to reduce it to a first derivative:

$$\int_I kAT''v\,dx = \left[kAT'v\right]_I - \int_I kAT'v'\,dx \tag{14}$$

Substituting back into the weak formulation, we get:

$$\left[kAT'v\right]_I - \int_I kAT'v'\,dx + \int_I hPTv\,dx = 0 \tag{15}$$

The term $[kAT'v]_I$ represents the boundary flux at the endpoints of the element.
For a quadratic approximation (P=2), we introduce three basis functions ϕ_1, ϕ_2, and ϕ_3:

$$\phi_1 = 1 - \frac{x}{\Delta x}, \quad \phi_2 = \frac{x}{\Delta x}, \quad \phi_3 = \phi_1 \cdot \phi_2 = \frac{x}{\Delta x} - \frac{x^2}{\Delta x^2}$$

with derivatives:

$$\phi_1' = -\frac{1}{\Delta x}, \quad \phi_2' = \frac{1}{\Delta x}, \quad \phi_3' = \frac{1}{\Delta x} - \frac{2x}{\Delta x^2}$$

Using these basis functions, we approximate $T(x)$ and $T'(x)$ as follows:

$$T(x) = T_{i-1}\phi_1 + T_i\phi_2 + \Omega\phi_3 \tag{16}$$

$$T'(x) = T_{i-1}\phi_1' + T_i\phi_2' + \Omega\phi_3' \tag{17}$$

Substituting these expressions into the weak formulation, we get:

$$-k_iA\int_I \left(T_{i-1}\phi_1'\phi_m' + T_i\phi_2'\phi_m' + \Omega\phi_3'\phi_m'\right)dx + hP\int_I \left(T_{i-1}\phi_1\phi_m + T_i\phi_2\phi_m + \Omega\phi_3\phi_m\right)dx = -\dot{Q}_i\phi_m \tag{18}$$

Define the stiffness coefficients k_{mn}^i:

$$k_{mn}^i = k_iA_i\int_0^{\Delta x}\phi_m'\phi_n'\,dx + hP_i\int_0^{\Delta x}\phi_m\phi_n\,dx \tag{19}$$

Using these definitions, we expand the system as follows:

$$\begin{bmatrix} k_{11}^i & k_{12}^i & k_{13}^i \\ k_{21}^i & k_{22}^i & k_{23}^i \\ k_{31}^i & k_{32}^i & k_{33}^i \end{bmatrix}\begin{bmatrix} T_{i-1} \\ T_i \\ \Omega_i \end{bmatrix} = \begin{bmatrix} -\dot{Q}_{i-1} \\ \dot{Q}_i \\ 0 \end{bmatrix} \tag{20}$$

For the right side, the system matrix for the next element is similarly formulated as:

$$\begin{bmatrix} k_{11}^{i+1} & k_{12}^{i+1} & k_{13}^{i+1} \\ k_{21}^{i+1} & k_{22}^{i+1} & k_{23}^{i+1} \\ k_{31}^{i+1} & k_{32}^{i+1} & k_{33}^{i+1} \end{bmatrix}\begin{bmatrix} T_i \\ T_{i+1} \\ \Omega_{i+1} \end{bmatrix} = \begin{bmatrix} -\dot{Q}_i \\ \dot{Q}_{i+1} \\ 0 \end{bmatrix} \tag{21}$$

To ensure continuity of heat flux across each internal node, we apply:

$$-\dot{Q}_i^+ + \dot{Q}_i^- = 0 \tag{22}$$

Expanding this equation with all terms, we have:

$$-k_{12}^iT_{i-1} + \left(k_{22}^i + k_{11}^{i+1}\right)T_i - k_{21}^{i+1}T_{i+1} + k_{13}^i\Omega_i + k_{23}^{i+1}\Omega_{i+1} = 0 \tag{23}$$

Additionally, we enforce the equation for Ω from each element:

$$k_{31}^iT_{i-1} + k_{32}^iT_i + k_{33}^i\Omega_i = 0 \tag{24}$$

Expanding for both elements on either side, we get:

$$k_{31}^{i}T_{i-1} + k_{32}^{i}T_i + k_{33}^{i}\Omega_i + k_{31}^{i+1}T_i + k_{32}^{i+1}T_{i+1} + k_{33}^{i+1}\Omega_{i+1} = 0 \tag{25}$$

The global matrix for a $P = 2$ FEM system with N nodes and $N-1$ elements is:

$$\begin{bmatrix}
k_{22}^1 + k_{11}^2 & k_{12}^2 & 0 & \cdots & 0 & k_{13}^2 & 0 & \cdots & 0 \\
k_{21}^2 & k_{22}^2 + k_{11}^3 & k_{12}^3 & \cdots & 0 & k_{23}^2 & k_{13}^3 & \cdots & 0 \\
0 & k_{21}^3 & k_{22}^3 + k_{11}^4 & \cdots & 0 & 0 & k_{23}^3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & k_{22}^{N-2} + k_{11}^{N-1} & 0 & 0 & \cdots & k_{13}^{N-1} \\
0 & 0 & 0 & \cdots & k_{21}^{N-1} & k_{22}^{N-1} & 0 & \cdots & k_{23}^{N-1} \\
k_{31}^1 & k_{32}^1 & 0 & \cdots & 0 & k_{33}^1 & 0 & \cdots & 0 \\
0 & k_{31}^2 & k_{32}^2 & \cdots & 0 & 0 & k_{33}^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & k_{31}^{N-1} & k_{32}^{N-1} & 0 & \cdots & k_{33}^{N-1}
\end{bmatrix}
\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_{N-2} \\ T_{N-1} \\ \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_{N-1} \end{bmatrix}
=
\begin{bmatrix} -k_{21}^1 T_0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -k_{31}^1 T_0 \\ 0 \\ \vdots \\ -k_{32}^N T_N \end{bmatrix}$$

Each row in the global matrix corresponds to the balance of temperature and flux terms, with the inclusion of Ω terms where appropriate. The top section of the matrix represents the temperature equations for each internal node, while the bottom section represents the equations for Ω terms for each element.

For a linear approximation ($P = 1$), the system is simplified because there are no Ω terms. The global matrix for $P = 1$ FEM with N nodes and $N-1$ elements is:

$$\begin{bmatrix}
k_{22}^1 + k_{11}^2 & k_{12}^2 & 0 & \cdots & 0 \\
k_{21}^2 & k_{22}^2 + k_{11}^3 & k_{12}^3 & \cdots & 0 \\
0 & k_{21}^3 & k_{22}^3 + k_{11}^4 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & k_{22}^{N-2} + k_{11}^{N-1}
\end{bmatrix}
\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_{N-1} \end{bmatrix}
=
\begin{bmatrix} -k_{21}^1 T_0 \\ 0 \\ 0 \\ \vdots \\ -k_{32}^N T_N \end{bmatrix}$$

Applying boundary conditions by setting known values for T_0 and T_N, we obtain a solvable linear system that yields the temperature distribution across the bar. Each row in this matrix represents the temperature balance for each node, with the system solvable using boundary conditions at T_0 and T_N.

Using the integrals and basis functions, the analytical values for k_{mn} terms are as follows:

$$k_{11}^i = \frac{k_i A_i}{\Delta x} + \frac{hP_i\Delta x}{3}$$

$$k_{12}^i = k_{21}^i = -\frac{k_i A_i}{\Delta x} + \frac{hP_i\Delta x}{6}$$

$$k_{22}^i = \frac{k_i A_i}{\Delta x} + \frac{hP_i\Delta x}{3}$$

$$k_{13}^i = k_{31}^i = \frac{hP_i\Delta x}{12}$$

$$k_{23}^i = k_{32}^i = \frac{hP_i\Delta x}{12}$$

$$k_{33}^{i} = \frac{hP_i\Delta x}{30}$$

These coefficients are derived based on the integrals of the respective basis functions and their derivatives, accounting for the material properties and boundary conditions.

3.3 Interface Temperature Results

3.3.1 Interface Location 1: $\bar{x} = 0.5$

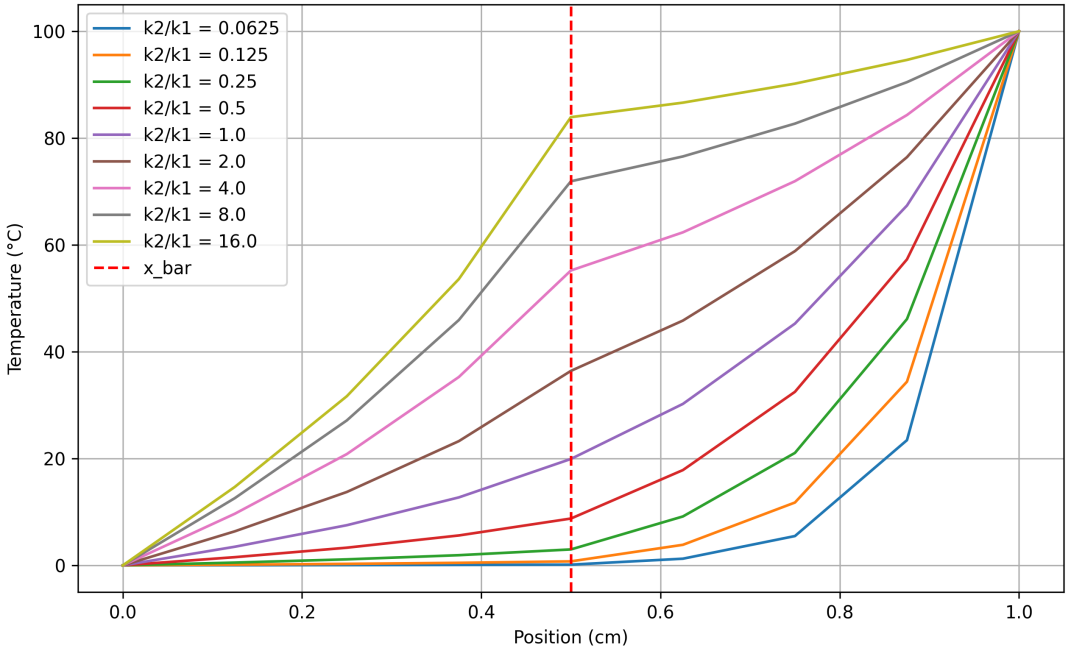

Figure 11: P=1 FEM Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

Table 11: P=1 FEM Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

	$\frac{k_2}{k_1}$								
x	0.0625	0.125	0.25	0.5	1.0	2.0	4.0	8.0	16.0
0.000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.125	0.008040	0.09242	0.4619	1.463	3.408	6.282	9.556	12.46	14.55
0.250	0.01737	0.1997	0.9980	3.161	7.363	13.57	20.65	26.93	31.44
0.375	0.02949	0.3389	1.694	5.366	12.50	23.04	35.05	45.71	53.37
0.500	0.04633	0.5326	2.662	8.432	19.64	36.20	55.07	71.82	83.87
0.625	0.4317	2.748	8.198	17.20	29.85	45.61	62.23	76.49	86.59
0.750	2.667	9.301	19.45	31.64	44.85	58.64	71.83	82.66	90.17
0.875	16.33	30.54	44.28	56.50	67.05	76.30	84.26	90.44	94.62
1.000	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

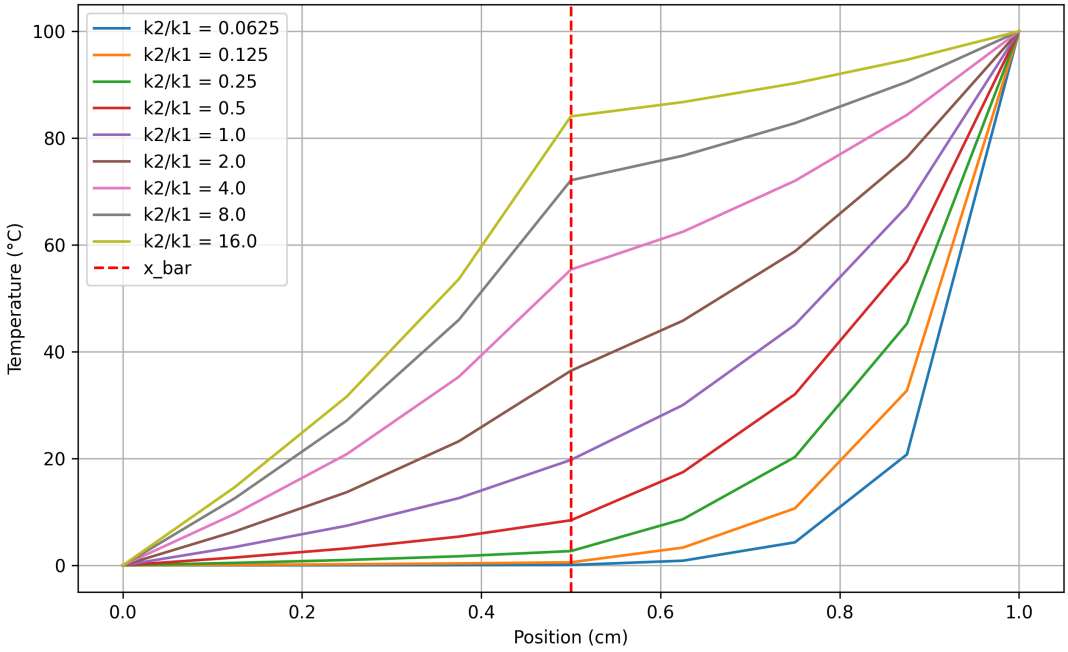

Figure 12: P=2 FEM Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

Table 12: P=2 FEM Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

	$\frac{k_2}{k_1}$								
x	0.0625	0.125	0.25	0.5	1.0	2.0	4.0	8.0	16.0
0.000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.125	0.01207	0.09822	0.4653	1.471	3.441	6.352	9.653	12.57	14.65
0.250	0.02605	0.2120	1.004	3.175	7.426	13.71	20.83	27.13	31.63
0.375	0.04415	0.3593	1.702	5.382	12.59	23.24	35.31	45.98	53.60
0.500	0.06924	0.5635	2.669	8.440	19.74	36.44	55.38	72.11	84.07
0.625	0.8674	3.316	8.632	17.44	30.02	45.82	62.46	76.70	86.74
0.750	4.294	10.65	20.27	32.03	45.04	58.79	71.99	82.80	90.26
0.875	20.73	32.72	45.25	56.89	67.20	76.39	84.34	90.51	94.67
1.000	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

3.3.2 Interface Location 2: $\bar{x} = \frac{2}{pi}$

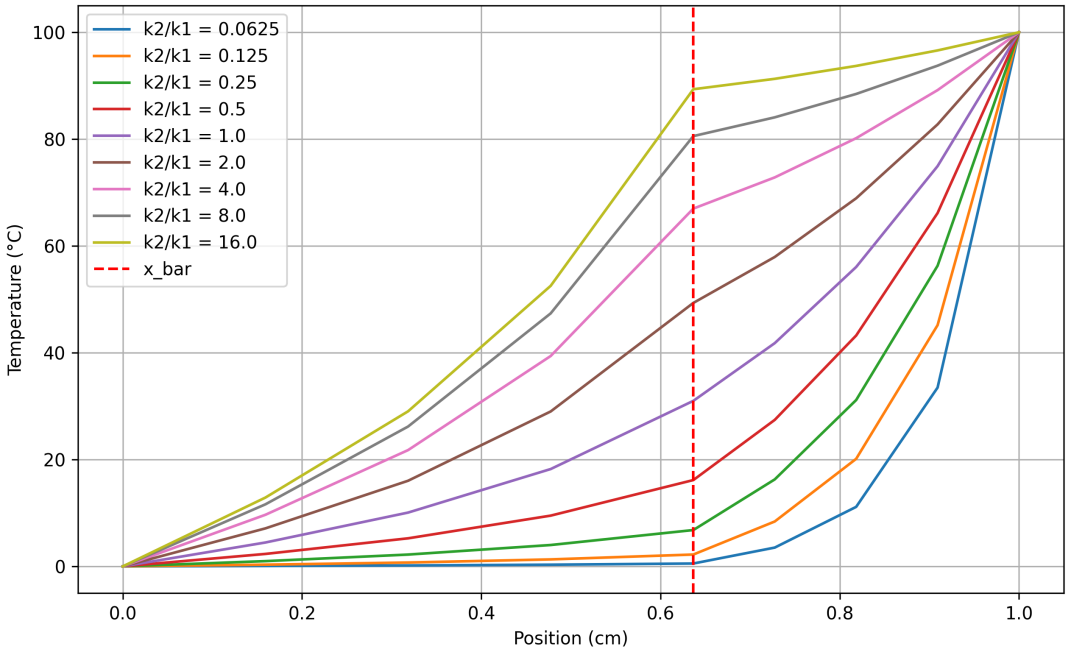

Figure 13: P=1 FEM Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

Table 13: P=1 FEM Temperature Distribution for Varying $\frac{k_2}{k_1}$ with $\bar{x} = \frac{2}{\pi}$

	$\frac{k_2}{k_1}$								
x	0.0625	0.125	0.25	0.5	1.0	2.0	4.0	8.0	16.0
0.000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.125	0.05187	0.2759	0.9180	2.241	4.339	6.933	9.434	11.36	12.60
0.250	0.1175	0.6247	2.079	5.075	9.826	15.70	21.36	25.72	28.54
0.375	0.2141	1.139	3.789	9.251	17.91	28.62	38.94	46.89	52.03
0.500	0.3674	1.954	6.502	15.87	30.73	49.11	66.82	80.45	89.27
0.625	2.435	7.452	15.65	27.04	41.54	57.74	72.70	83.99	91.23
0.750	8.624	18.48	30.26	42.79	55.83	68.78	80.08	88.39	93.67
0.875	29.41	43.21	55.45	65.81	74.79	82.67	89.12	93.71	96.58
1.000	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

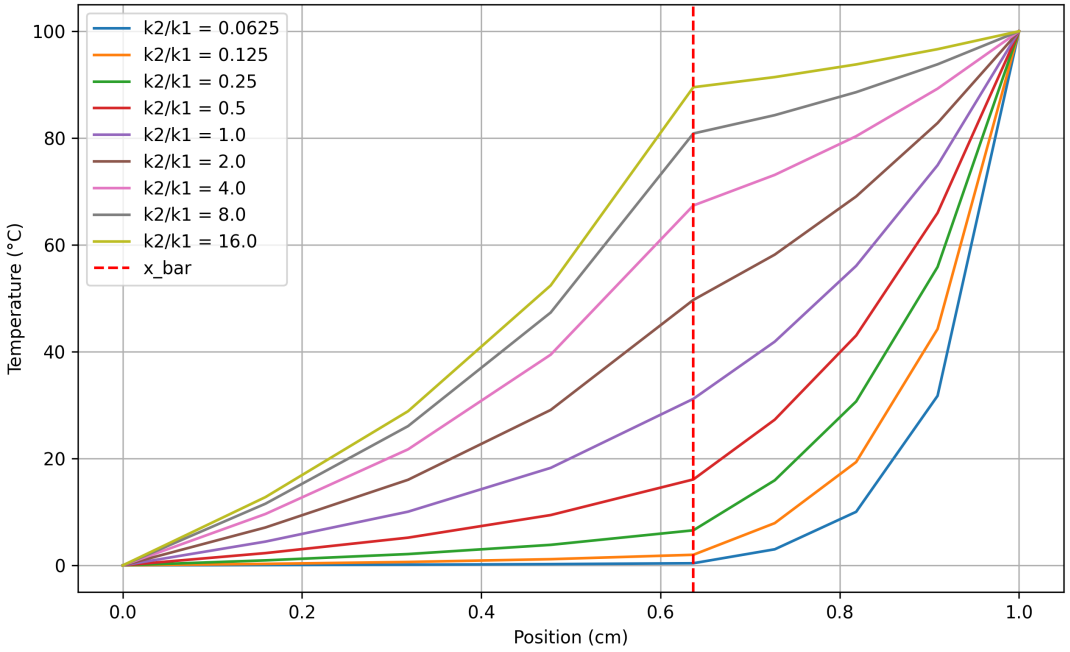

Figure 14: P=2 FEM Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

Table 14: P=2 FEM Temperature Distribution for Varying $\frac{k_2}{k_1}$ with $\bar{x} = \frac{2}{\pi}$

	$\frac{k_2}{k_1}$								
x	0.0625	0.125	0.25	0.5	1.0	2.0	4.0	8.0	16.0
0.000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.125	0.05636	0.2821	0.9367	2.2967	4.451	7.095	9.619	11.54	12.78
0.250	0.1273	0.6371	2.115	5.187	10.05	16.03	21.73	26.07	28.86
0.375	0.2312	1.157	3.842	9.420	18.26	29.10	39.45	47.34	52.42
0.500	0.3948	1.976	6.562	16.09	31.18	49.71	67.38	80.86	89.52
0.625	3.001	7.903	15.93	27.28	41.89	58.18	73.11	84.29	91.42
0.750	10.01	19.34	30.69	43.04	56.08	69.07	80.35	88.60	93.79
0.875	31.72	44.25	55.87	66.00	74.93	82.82	89.25	93.81	96.65
1.000	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

3.3.3 Discussion

From the results in Figure 11, Figure 12, Figure 13, and Figure 14, both FEM results for both cases showed similar results to the analytical temperature distribution in Figure 1 and Figure 2.

3.4 Heat Convection Results

3.4.1 Interface Location 1: $\bar{x} = 0.5$

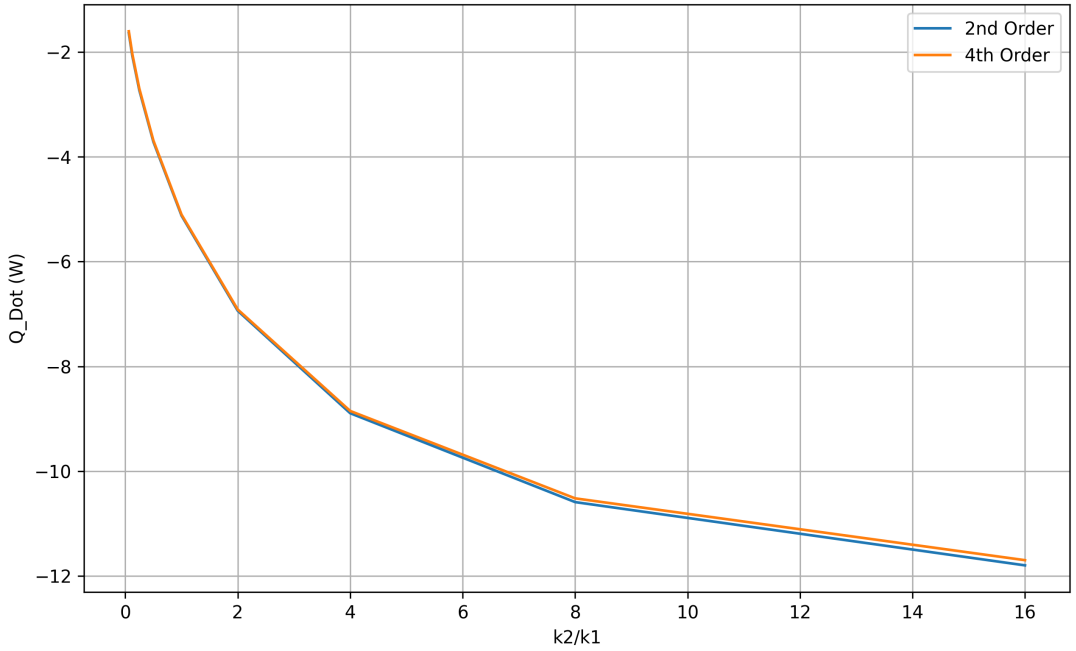

Figure 15: P=1 and P=2 FEM Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

Table 15: P=1 and P=2 FEM Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

$\frac{k_2}{k_1}$	2nd Order FDM Q	4th Order FDM Q
0.0625	-1.6389	-1.6043
0.125	-2.0728	-2.0386
0.25	-2.7322	-2.7017
0.5	-3.7146	-3.6902
1.0	-5.1226	-5.1031
2.0	-6.9387	-6.9149
4.0	-8.8940	-8.8513
8.0	-10.590	-10.519
16.0	-11.798	-11.699

3.4.2 Interface Location 2: $\bar{x} = \frac{2}{pi}$

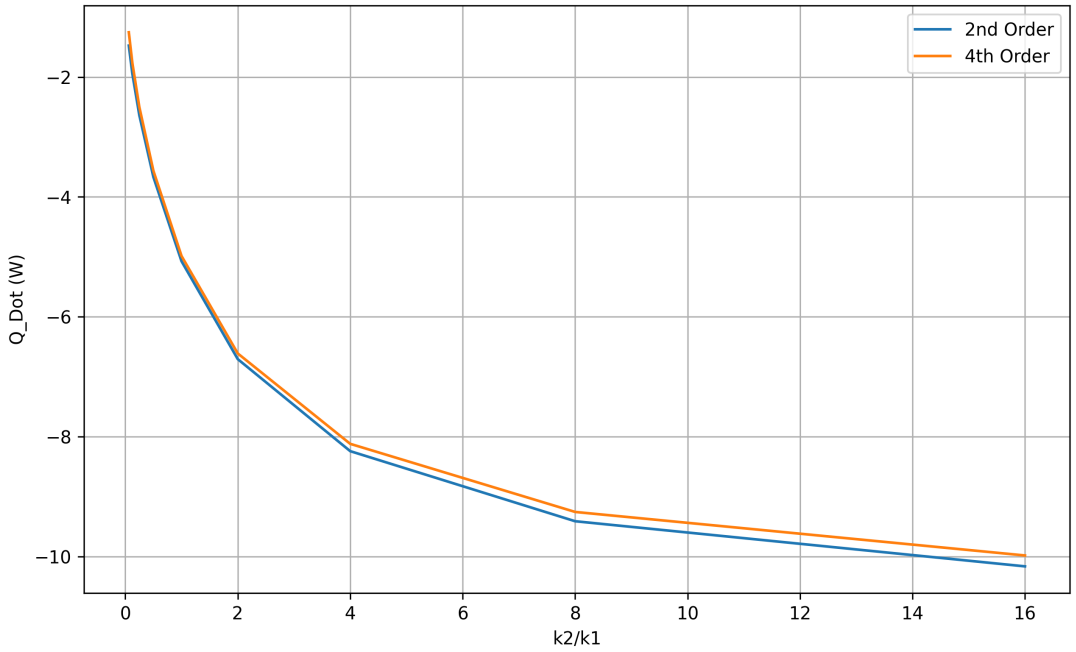

Figure 16: P=1 and P=2 FEM Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

Table 16: P=1 and P=2 FEM Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

$\frac{k_2}{k_1}$	2nd Order FDM Q	4th Order FDM Q
0.0625	-1.4763	-1.2539
0.125	-1.9409	-1.7636
0.25	-2.6394	-2.5031
0.5	-3.6692	-3.5649
1.0	-5.0734	-4.9854
2.0	-6.7058	-6.6110
4.0	-8.2408	-8.1189
8.0	-9.4097	-9.2548
16.0	-10.163	-9.9808

3.4.3 Discussion

Looking at the heat convection results in Figure 15 and Figure 16 compared to Figure 3 and Figure 4 show similar heat convection results at the end of the bar for both methods and interface locations.

4 FDM and FEM Convergence

4.1 Abstract

The convergence of the FDM and FEM are important quantities when verifying the method and predicting future performance for different setups. The rate of convergence of both methods will be calculated in reference to the exact quantity and the quantity found through Richardson Extrapolation.
The convergence of the interface temperature and the heat convection at the right end of the rod will be analyzed.

4.2 Interface Temperature Convergence

4.2.1 Interface Location 1: $\bar{x} = 0.5$

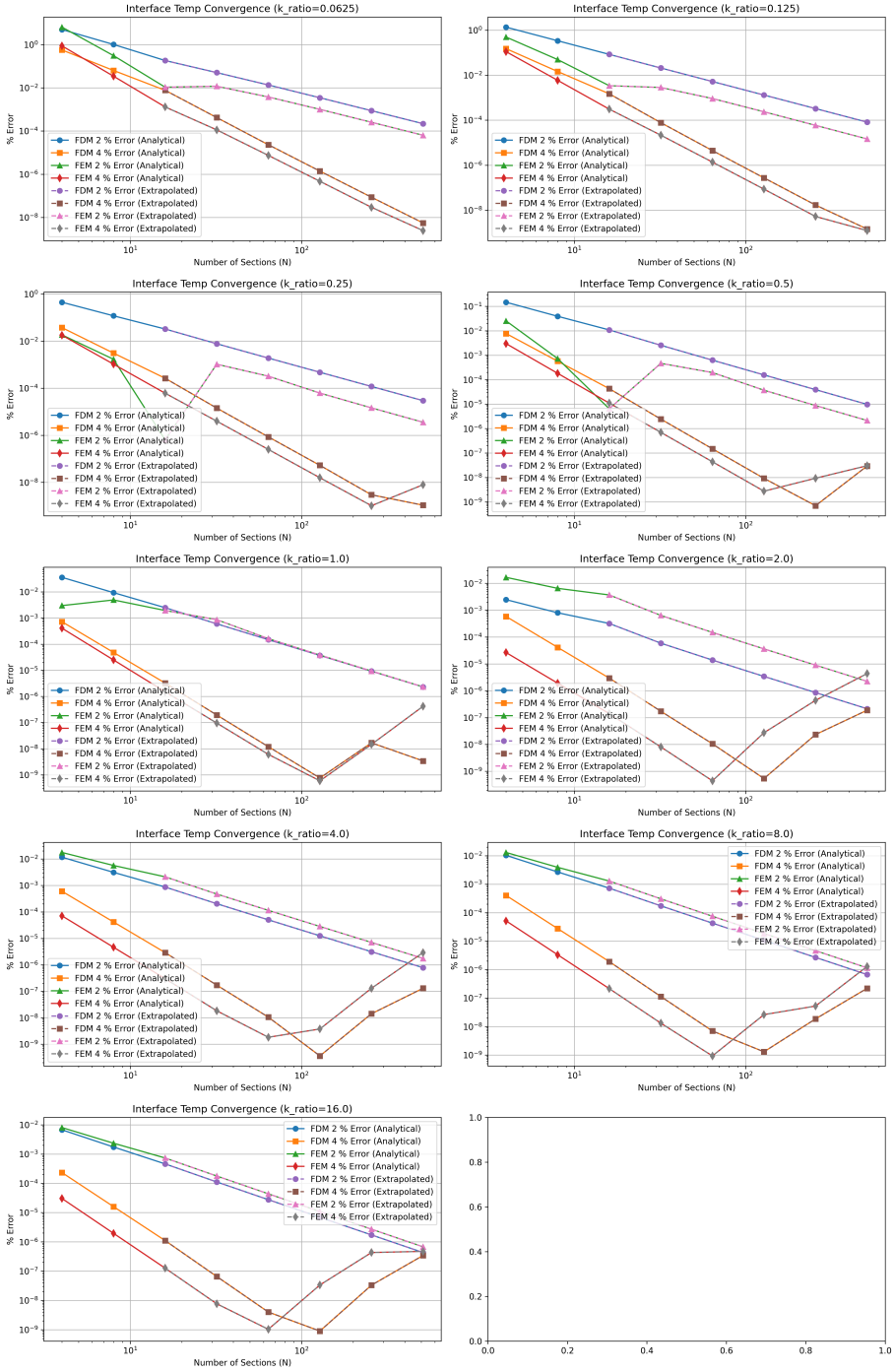

Figure 17: FDM and FEM Temperature Convergence for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

Table 17: 2nd Order FDM Temperature Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = 0.5$

N	True T	T	T_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	55.3837	54.7385	-	1.17e-2	-	-	-
8	55.3837	55.2113	-	3.11e-3	-1.90	-	-
16	55.3837	55.3399	55.3879	8.67e-4	-1.97	8.67e-4	-
32	55.3837	55.3727	55.3840	2.04e-4	-1.99	2.04e-4	-1.97
64	55.3837	55.3810	55.3838	5.01e-5	-1.99	5.01e-5	-1.99
128	55.3837	55.3831	55.3837	1.25e-5	-2.00	1.25e-5	-2.00
256	55.3837	55.3836	55.3837	3.11e-6	-2.00	3.11e-6	-2.00
512	55.3837	55.3837	55.3837	7.78e-7	-2.00	7.78e-7	-2.00

Table 18: 4th FDM Order Temperature Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = 0.5$

N	True T	T	T_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	55.3837	55.3505	-	6.00e-4	-	-	-
8	55.3837	55.3815	-	4.14e-5	-3.86	-	-
16	55.3837	55.3836	55.3838	2.89e-6	-3.96	2.89e-6	-
32	55.3837	55.3837	55.3837	1.71e-7	-3.99	1.71e-7	-3.96
64	55.3837	55.3837	55.3837	1.05e-8	-4.00	1.05e-8	-3.99
128	55.3837	55.3837	55.3837	3.60e-10	-3.99	3.60e-10	-4.82
256	55.3837	55.3837	55.3837	1.41e-8	-4.01	1.41e-8	0.0640
512	55.3837	55.3837	55.3837	1.30e-7	-4.03	1.30e-7	4.24e-4

Table 19: P=1 FEM Temperature Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = 0.5$

N	True T	T	T_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	55.3837	54.4102	-	1.76e-2	-	-	-
8	55.3837	55.0701	-	5.66e-3	-1.63	-	-
16	55.3837	55.2956	55.4126	2.11e-3	-1.83	2.11e-3	-
32	55.3837	55.3604	55.3866	4.72e-4	-1.92	4.72e-4	-1.80
64	55.3837	55.3778	55.3841	1.14e-4	-1.96	1.14e-4	-1.90
128	55.3837	55.3822	55.3838	2.80e-5	-1.98	2.80e-5	-1.95
256	55.3837	55.3834	55.3838	7.00e-6	-1.99	7.00e-6	-1.98
512	55.3837	55.3837	55.3837	2.00e-6	-2.00	2.00e-6	-1.99

Table 20: P=2 FEM Temperature Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = 0.5$

N	True T	T	T_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	55.3837	55.3798	-	7.08e-5	-	-	-
8	55.3837	55.3835	-	4.59e-6	-3.95	-	-
16	55.3837	55.3837	55.3837	2.99e-7	-3.99	2.99e-7	-
32	55.3837	55.3837	55.3837	1.83e-8	-4.00	1.83e-8	-3.99
64	55.3837	55.3837	55.3837	1.83e-9	-4.00	1.83e-9	-3.37
128	55.3837	55.3837	55.3837	3.75e-9	-3.99	3.75e-9	-0.36
256	55.3837	55.3837	55.3837	1.30e-7	-3.95	1.30e-7	0.741
512	55.3837	55.3837	55.3839	2.90e-6	-1.10	2.90e-6	-3.00e-6

All other tables for their respective $\frac{k2}{k1}$ value are given in the Raw Code Output Appendix due to the sheer volume of data output.

4.2.2 Interface Location 2: $\bar{x} = \frac{2}{pi}$

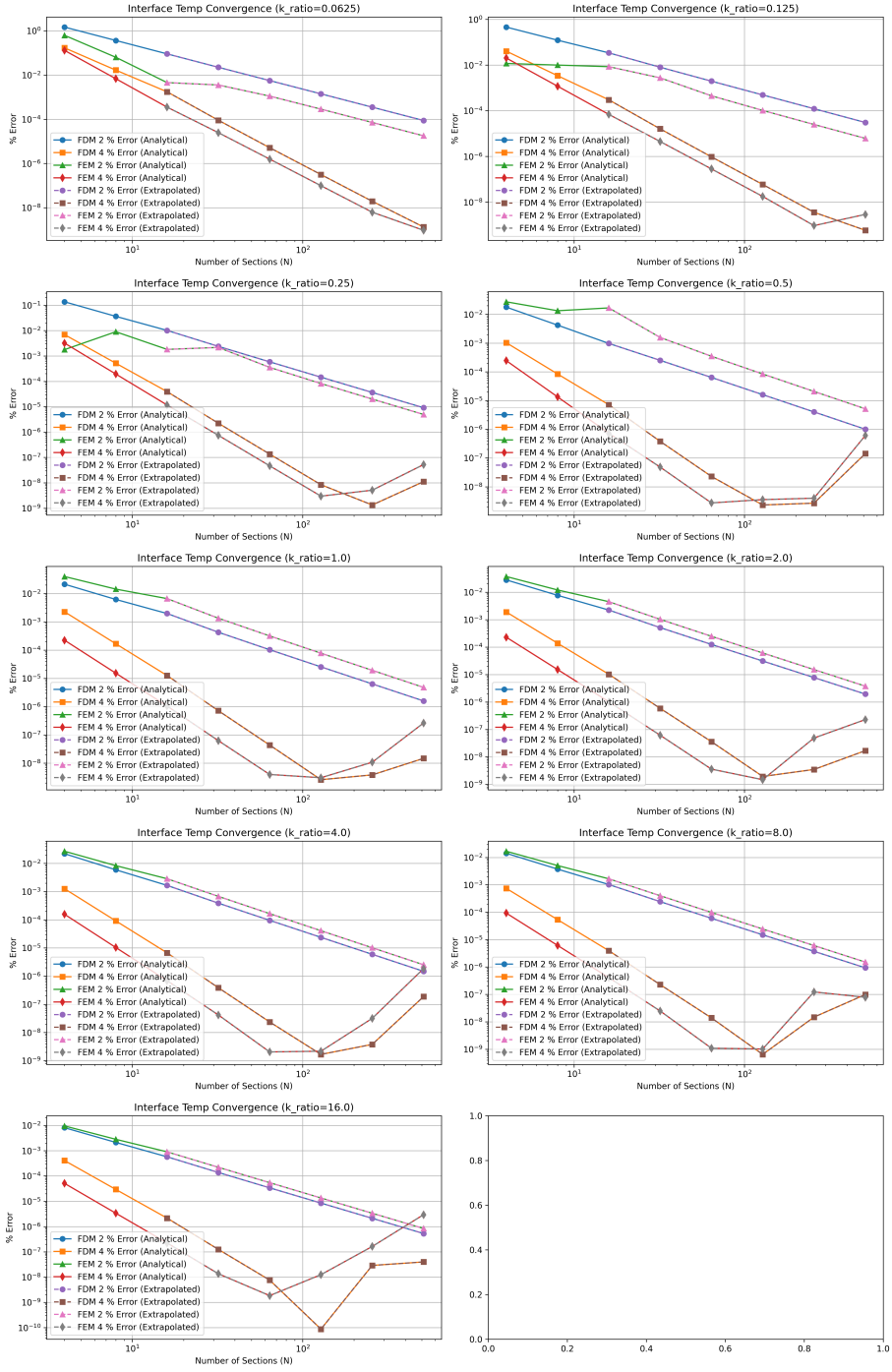

Figure 18: FDM and FEM Temperature Convergence for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

Table 21: 2nd Order FDM Temperature Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = \frac{2}{\pi}$

N	True T	T	T_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	67.3849	65.8951	-	2.21e-2	-	-	-
8	67.3849	66.9854	-	5.93e-3	-1.90	-	-
16	67.3849	67.2831	67.3949	1.66e-3	-1.97	1.66e-3	-
32	67.3849	67.3593	67.3855	3.89e-4	-1.99	3.89e-4	-1.97
64	67.3849	67.3785	67.3849	9.59e-5	-2.00	9.59e-5	-1.99
128	67.3849	67.3833	67.3849	2.37e-5	-2.00	2.37e-5	-1.99
256	67.3849	67.3845	67.3849	5.95e-6	-2.00	5.95e-6	-2.00
512	67.3849	67.3848	67.3849	1.49e-6	-2.00	1.49e-6	-2.00

Table 22: 4th Order FDM Temperature Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = \frac{2}{\pi}$

N	True T	T	T_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	67.3849	67.2997	-	1.26e-3	-	-	-
8	67.3849	67.3787	-	9.13e-5	-3.79	-	-
16	67.3849	67.3845	67.3849	6.71e-6	-3.94	6.71e-6	-
32	67.3849	67.3848	67.3849	3.87e-7	-3.99	3.87e-7	-3.94
64	67.3849	67.3849	67.3849	2.37e-8	-4.00	2.37e-8	-3.98
128	67.3849	67.3849	67.3849	1.67e-9	-4.00	1.67e-9	-3.82
256	67.3849	67.3849	67.3849	3.79e-9	-3.98	3.79e-9	6.50e-1
512	67.3849	67.3849	67.3849	1.87e-7	-4.20	1.87e-7	-6.76e-4

Table 23: P=2 FEM Temperature Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = \frac{2}{\pi}$

N	True T	T	T_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	67.3849	65.5624	-	2.70e-2	-	-	-
8	67.3849	66.8226	-	8.34e-3	-1.70	-	-
16	67.3849	67.2284	67.4212	2.86e-3	-1.85	2.86e-3	-
32	67.3849	67.3436	67.3892	6.77e-4	-1.92	6.77e-4	-1.82
64	67.3849	67.3743	67.3854	1.65e-4	-1.96	1.65e-4	-1.91
128	67.3849	67.3822	67.3849	4.10e-5	-1.98	4.10e-5	-1.95
256	67.3849	67.3842	67.3849	1.00e-5	-1.99	1.00e-5	-1.98
512	67.3849	67.3847	67.3849	2.97e-6	-2.00	2.97e-6	-1.99

Table 24: P=2 FEM Temperature Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = \frac{2}{\pi}$

N	True T	T	T_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	67.3849	67.3743	-	1.57e-4	-	-	-
8	67.3849	67.3842	-	1.04e-5	-3.92	-	-
16	67.3849	67.3848	67.3849	6.91e-7	-3.98	6.91e-7	-
32	67.3849	67.3849	67.3849	4.19e-8	-3.99	4.19e-8	-3.98
64	67.3849	67.3849	67.3849	2.05e-9	-4.00	2.05e-9	-4.32
128	67.3849	67.3849	67.3849	2.17e-9	-4.00	2.17e-9	3.11
256	67.3849	67.3849	67.3849	3.18e-8	-4.07	3.18e-8	6.92e-3
512	67.3849	67.3849	67.3850	1.85e-6	-5.91	1.85e-6	-7.00e-6

All other tables for their respective $\frac{k2}{k1}$ value are given in the Raw Code Output Appendix due to the sheer volume of data output.

4.2.3 Discussion

Referencing Figure 17 and Figure 18, 2nd order FDM and P=1 FEM converge similar with a $= 2$ rate. This is true of the convergence in reference to the exact temperature at the interface and the extrapolated value. Similarly, the 4th order FDM and P=2 FEM converge at $= 4$ rate.

Interestingly, it can be noted that the convergence in reference to the extrapolated values when using 4th order FDM and p=2 FEm appear to diverge past $N = 256$. This can be attributed to machine precision, where calculating the extrapolated values deal with too small of numbers for Python to represent with single precision. If care is taken to artificially increase the values in the computation, or represent the values in a method other than floating point, the expected convergence would resume.

4.3 Heat Convection Convergence

4.3.1 Interface Location 1: $\bar{x} = 0.5$

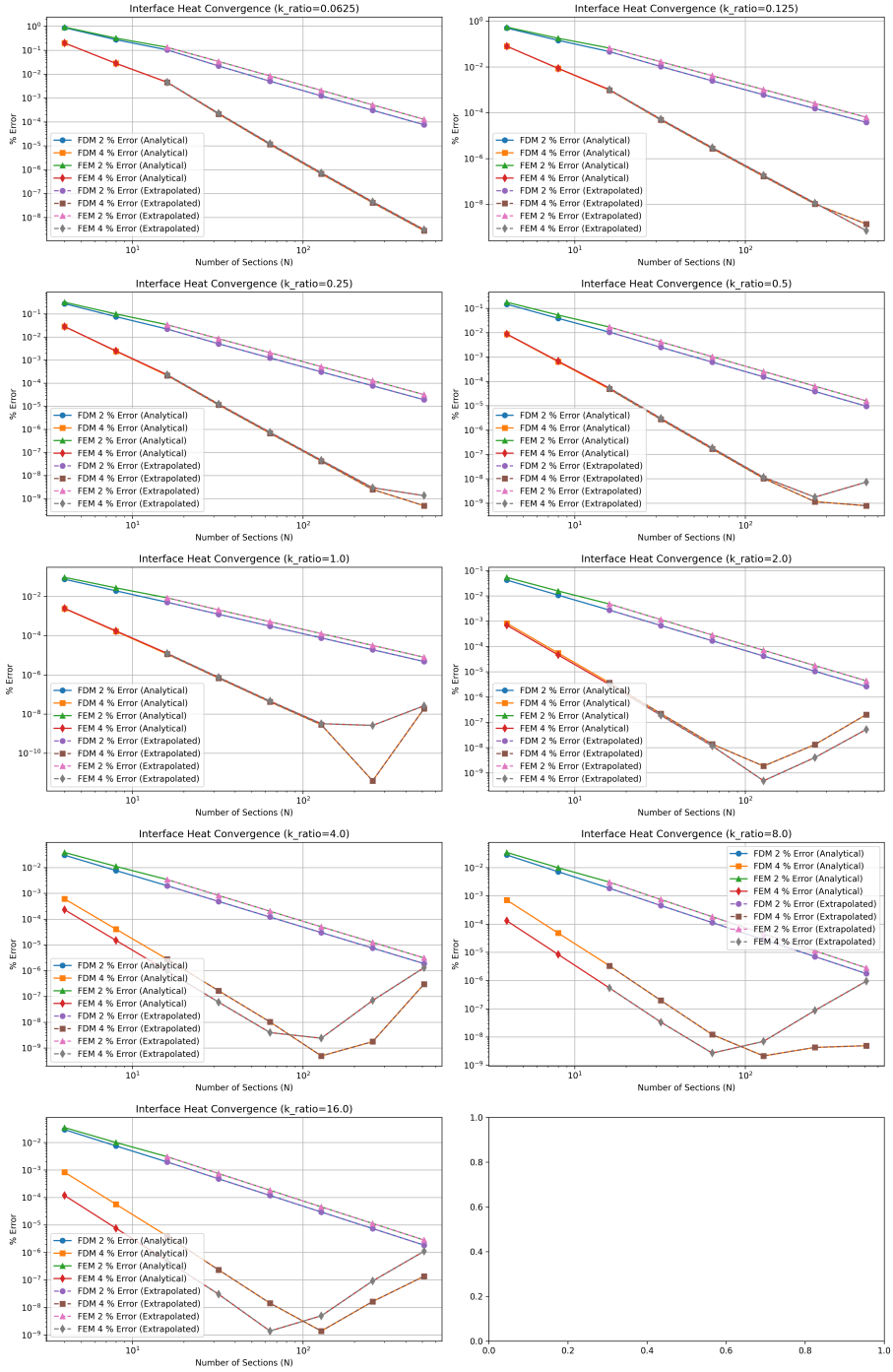

Figure 19: FDM and FEM Heat Convection Convergence for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

Table 25: FDM Heat Convection Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = 0.5$

N	True Q	Q	Q_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	-8.7970	-9.0603	N/A	0.029924	N/A	N/A	N/A
8	-8.7970	-8.8640	N/A	0.007612	1.97486	N/A	N/A
16	-8.7970	-8.8139	-8.7967	0.001912	1.99334	0.001956	1.96860
32	-8.7970	-8.8013	-8.7970	0.000479	1.99831	0.000481	1.99167
64	-8.7970	-8.7981	-8.7970	0.000120	1.99958	0.000120	1.99789
128	-8.7970	-8.7973	-8.7970	0.000030	1.99989	0.000030	1.99947

Table 26: FEM Heat Convection Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = 0.5$

N	True Q	Q	Q_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	-8.7970	-8.9466	N/A	0.017004	N/A	N/A	N/A
8	-8.7970	-8.8425	N/A	0.005169	1.7180	N/A	N/A
16	-8.7970	-8.8095	-8.7942	0.001419	1.8653	0.001740	1.6581
32	-8.7970	-8.8003	-8.7967	0.000371	1.9340	0.000406	1.8401
64	-8.7970	-8.7979	-8.7970	0.000095	1.9673	0.000099	1.9224
128	-8.7970	-8.7973	-8.7970	0.000024	1.9838	0.000025	1.9617

All other tables for their respective $\frac{k2}{k1}$ value are given in the Raw Code Output Appendix due to the sheer volume of data output.

4.3.2 Interface Location 2: $\bar{x} = \frac{2}{pi}$

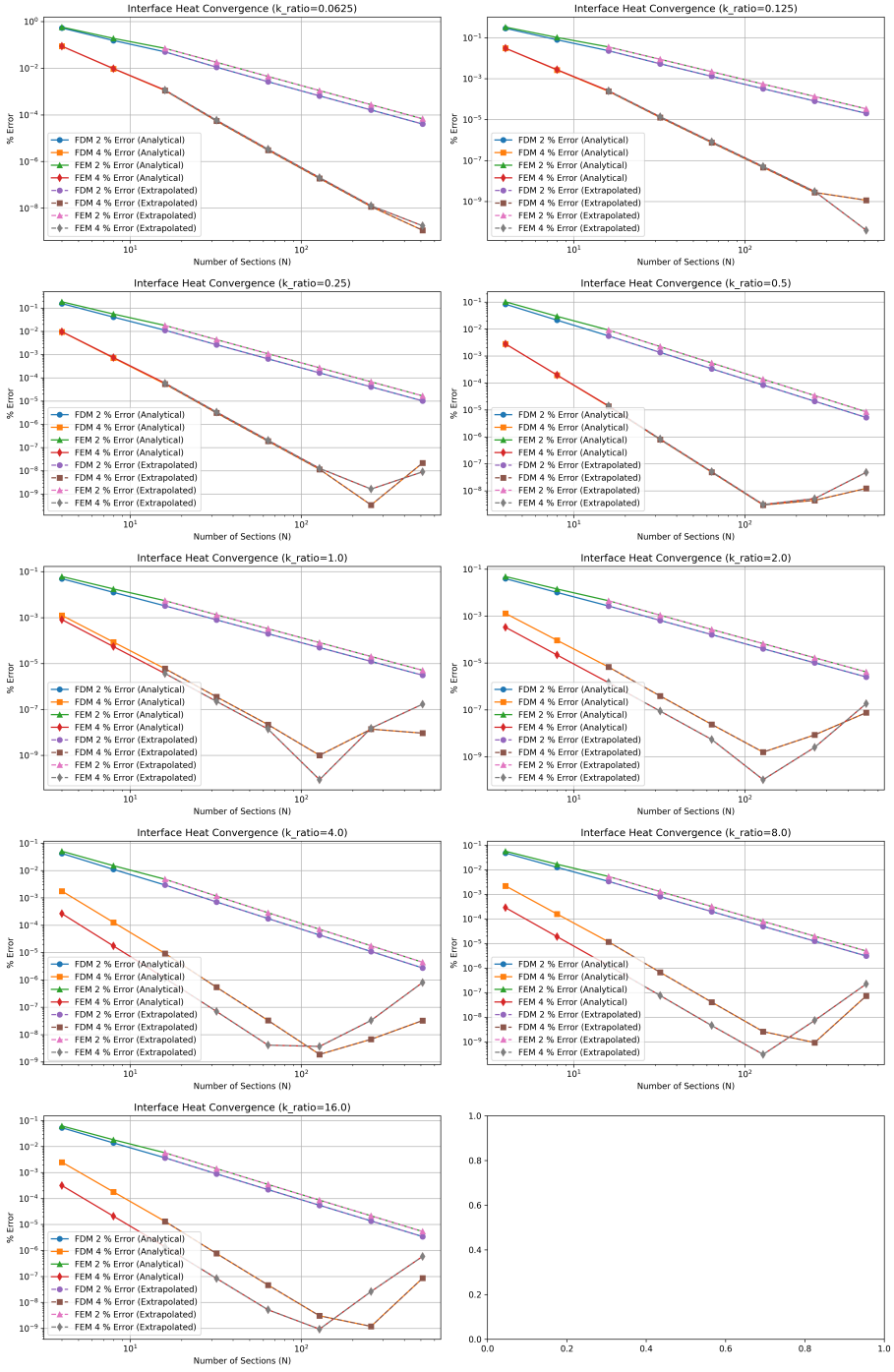

Figure 20: FDM and FEM Heat Convection Convergence for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

Table 27: FDM Heat Convection Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = \frac{2}{\pi}$

N	True Q	Q	Q_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	-8.1187	-8.4608	N/A	0.042135	N/A	N/A	N/A
8	-8.1187	-8.2081	N/A	0.011009	1.93636	N/A	N/A
16	-8.1187	-8.1413	-8.1174	0.002786	1.98260	0.002953	1.92036
32	-8.1187	-8.1244	-8.1186	0.000699	1.99554	0.000710	1.97824
64	-8.1187	-8.1201	-8.1187	0.000175	1.99888	0.000175	1.99443
128	-8.1187	-8.1191	-8.1187	0.000044	1.99972	0.000044	1.99860

Table 28: FEM Heat Convection Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = \frac{2}{\pi}$

N	True Q	Q	Q_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	-8.1187	-8.4335	N/A	0.038775	N/A	N/A	N/A
8	-8.1187	-8.2149	N/A	0.011850	1.71028	N/A	N/A
16	-8.1187	-8.1454	-8.1129	0.003283	1.85197	0.004001	1.65208
32	-8.1187	-8.1257	-8.1180	0.000864	1.92490	0.000951	1.82497
64	-8.1187	-8.1205	-8.1186	0.000222	1.96214	0.000233	1.91181
128	-8.1187	-8.1192	-8.1187	0.000056	1.98099	0.000058	1.95569

All other tables for their respective $\frac{k2}{k1}$ value are given in the Raw Code Output Appendix due to the sheer volume of data output.

4.3.3 Discussion

Referencing Figure 19 and Figure 20, 2nd order FDM and P=1 FEM converge similar with a $= 2$ rate. This is true of the convergence in reference to the exact heat flux at the end of the bar and the extrapolated value. Similarly, the 4th order FDM and P=2 FEM converge at $= 4$ rate.

Interestingly, it can be noted that the convergence in reference to the extrapolated values when using 4th order FDM and p=2 FEm appear to diverge past $N = 256$. This can be attributed to machine precision, where calculating the extrapolated values deal with too small of numbers for Python to represent with single precision. If care is taken to artificially increase the values in the computation, or represent the values in a method other than floating point, the expected convergence would resume.

5 Raw Code Output

```
Section 1: Analytical
Analytical Temperature Results x_bar = 0.5:
              x = 0.000  x = 0.125  x = 0.250  x = 0.375  x = 0.500  x = 0.500  x = 0.625  x = 0.750  x = 0.875  x = 1.000
k2/k1 = 0.0625       0.0   0.011667   0.025181   0.042681   0.066937   0.066937   0.847794   4.228179  20.573130      100.0
k2/k1 = 0.125        0.0   0.097641   0.210739   0.357195   0.560194   0.560194   3.303981  10.626282  32.674022      100.0
k2/k1 = 0.25         0.0   0.464798   1.003171   1.700341   2.666666   2.666666   8.626608  20.264910  45.242343      100.0
k2/k1 = 0.5          0.0   1.470842   3.174511   5.380689   8.438605   8.438605  17.437015  32.027881  56.890833      100.0
k2/k1 = 1.0          0.0   3.440413   7.425426  12.585848  19.738549  19.738549  30.015765  45.044332  67.203196      100.0
k2/k1 = 2.0          0.0   6.351689  13.708819  23.235988  36.441301  36.441301  45.815586  58.792577  76.392720      100.0
k2/k1 = 4.0          0.0   9.653341  20.834759  35.314219  55.383747  55.383747  62.464754  71.993744  84.344156      100.0
k2/k1 = 8.0          0.0  12.567937  27.125318  45.976504  72.105546  72.105546  76.701212  82.797389  90.513336      100.0
k2/k1 = 16.0         0.0  14.652807  31.625083  53.603455  84.066993  84.066993  86.741990  90.264767  94.669752      100.0

Analytical Temperature Results x_bar = 2/pi:
              x = 0.000  x = 0.125  x = 0.250  x = 0.375  x = 0.500  x = 0.625  x = 0.637  x = 0.750  x = 0.875  x = 1.000
k2/k1 = 0.0625       0.0   0.043265   0.093379   0.158275   0.248224   0.377466   0.392138   4.086217  20.545109      100.0
k2/k1 = 0.125        0.0   0.217770   0.470012   0.796654   1.249403   1.899926   1.973771  10.004385  32.490342      100.0
k2/k1 = 0.25         0.0   0.723845   1.562272   2.647999   4.152891   6.315166   6.560619  18.929061  44.739811      100.0
k2/k1 = 0.5          0.0   1.775153   3.831304   6.493931  10.184517  15.487261  16.089210  30.677503  56.308955      100.0
k2/k1 = 1.0          0.0   3.440413   7.425426  12.585848  19.738549  30.015765  31.182398  45.044332  67.203196      100.0
k2/k1 = 2.0          0.0   5.484175  11.836467  20.062415  31.464146  47.846498  49.706164  60.645947  77.284348      100.0
k2/k1 = 4.0          0.0   7.434698  16.046273  27.197894  42.654810  64.863775  67.384857  74.763593  85.702464      100.0
k2/k1 = 8.0          0.0   8.921798  19.255874  32.638061  51.186694  77.837932  80.863285  85.279652  91.742445      100.0
k2/k1 = 16.0         0.0   9.877475  21.318505  36.134152  56.669658  86.175696  89.525116  91.966805  95.516633      100.0

Analytical Heat Convection Results x_bar = 0.5:
              Heat Convection
k2/k1 = 0.0625       -1.241829
k2/k1 = 0.125        -1.756437
k2/k1 = 0.25         -2.486946
k2/k1 = 0.5          -3.529489
k2/k1 = 1.0          -4.985127
k2/k1 = 2.0          -6.832006
k2/k1 = 4.0          -8.797049
k2/k1 = 8.0         -10.486214
k2/k1 = 16.0        -11.680625

Analytical Heat Convection Results x_bar = 2/pi:
              Heat Convection
k2/k1 = 0.0625       -1.241978
k2/k1 = 0.125        -1.758799
k2/k1 = 0.25         -2.501202
k2/k1 = 0.5          -3.564213
k2/k1 = 1.0          -4.985127
k2/k1 = 2.0          -6.610807
k2/k1 = 4.0          -8.118713
k2/k1 = 8.0          -9.254592
k2/k1 = 16.0         -9.980639

Section 2: FDM with Varying k_ratio
FDM 2nd Order Temperature Results x_bar = 0.5:
              x = 0.000  x = 0.125  x = 0.250  x = 0.375  x = 0.500  x = 0.625  x = 0.750  x = 0.875  x = 1.000
k2/k1 = 0.0625       0.0   0.023566   0.050814   0.086002   0.134628   1.249213   5.486829  23.441518      100.0
k2/k1 = 0.125        0.0   0.131160   0.282813   0.478656   0.749289   3.850963  11.766340  34.389643      100.0
k2/k1 = 0.25         0.0   0.522237   1.126073   1.905858   2.983433   9.158380  21.057315  46.117073      100.0
k2/k1 = 0.5          0.0   1.534785   3.309379   5.601064   8.767916  17.841592  32.490766  57.293304      100.0
k2/k1 = 1.0          0.0   3.487383   7.519670  12.726906  19.922720  30.231460  45.263865  67.368749      100.0
k2/k1 = 2.0          0.0   6.373779  13.743460  23.260557  36.412116  45.832593  58.833740  76.431273      100.0
k2/k1 = 4.0          0.0   9.664493  20.839062  35.269736  55.211305  62.353389  71.931152  84.318726      100.0
k2/k1 = 8.0          0.0  12.587755  27.142347  45.937931  71.911316  76.562507  82.709060  90.471024      100.0
k2/k1 = 16.0         0.0  14.689868  31.675027  53.609410  83.920263  86.634225  90.194225  94.635027      100.0

FDM 4th Order Temperature Results x_bar = 0.5:
              x = 0.000  x = 0.125  x = 0.250  x = 0.375  x = 0.500  x = 0.625  x = 0.750  x = 0.875  x = 1.000
k2/k1 = 0.0625       0.0   0.012407   0.026778   0.045387   0.071181   0.872956   4.311784  20.775791      100.0
k2/k1 = 0.125        0.0   0.099063   0.213805   0.362390   0.568335   3.324588  10.669465  32.740427      100.0
k2/k1 = 0.25         0.0   0.466246   1.006291   1.705616   2.674914   8.638042  20.281132  45.260122      100.0
k2/k1 = 0.5          0.0   1.471712   3.176373   5.383805   8.443410  17.441951  32.033043  56.895148      100.0
k2/k1 = 1.0          0.0   3.440655   7.425912  12.586575  19.739499  30.016879  45.045466  67.204053      100.0
k2/k1 = 2.0          0.0   6.351568  13.708490  23.235255  36.439800  45.814649  58.792070  76.392524      100.0
k2/k1 = 4.0          0.0   9.653156  20.834256  35.313096  55.381453  62.463124  71.992703  84.343652      100.0
k2/k1 = 8.0          0.0  12.567869  27.125037  45.975677  72.103555  76.699755  82.796434  90.512864      100.0
k2/k1 = 16.0         0.0  14.652898  31.625124  53.603116  84.065651  86.740995  90.264109  94.669425      100.0

FDM 2nd Order Temperature Results x_bar = 2/pi:
              x = 0.000  x = 0.159  x = 0.318  x = 0.477  x = 0.637  x = 0.727  x = 0.818  x = 0.909  x = 1.000
k2/k1 = 0.0625       0.0   0.077055   0.173629   0.314183   0.534322   3.515595  11.139042  33.471059      100.0
k2/k1 = 0.125        0.0   0.319719   0.720423   1.303613   2.217011   8.401972  20.134133  45.159372      100.0
k2/k1 = 0.25         0.0   0.980497   2.209358   3.997855   6.779021  16.282888  31.141948  56.281369      100.0
k2/k1 = 0.5          0.0   2.330072   5.250358   9.500576  16.157318  27.426136  43.221817  66.151538      100.0
k2/k1 = 1.0          0.0   4.469024  10.070065  18.221884  30.989359  41.796025  56.052044  74.933939      100.0
k2/k1 = 2.0          0.0   7.113289  16.028395  29.003540  49.325368  57.925784  68.916457  82.750907      100.0
k2/k1 = 4.0          0.0   9.660074  21.767073  39.387737  66.985431  72.825251  80.167607  89.163985      100.0
k2/k1 = 8.0          0.0  11.617645  26.178075  47.369488  80.559732  84.071349  88.450249  93.741604      100.0
k2/k1 = 16.0         0.0  12.883057  29.029430  52.529043  89.334419  91.281472  93.699357  96.600545      100.0

FDM 4th Order Temperature Results x_bar = 2/pi:
              x = 0.000  x = 0.159  x = 0.318  x = 0.477  x = 0.637  x = 0.727  x = 0.818  x = 0.909  x = 1.000
k2/k1 = 0.0625       0.0   0.056906   0.128532   0.233402   0.398641   3.008594  10.028399  31.747344      100.0
k2/k1 = 0.125        0.0   0.282730   0.638587   1.159615   1.980577   7.908859  19.346064  44.258786      100.0
k2/k1 = 0.25         0.0   0.937019   2.116398   3.843183   6.564006  15.927669  30.693906  55.871337      100.0
k2/k1 = 0.5          0.0   2.296558   5.187121   9.419332  16.087851  27.280752  43.038453  65.997640      100.0
k2/k1 = 1.0          0.0   4.450576  10.052292  18.254031  31.177173  41.886144  56.075679  74.924868      100.0
k2/k1 = 2.0          0.0   7.094631  16.024287  29.098623  49.699312  58.180432  69.070572  82.820650      100.0
k2/k1 = 4.0          0.0   9.618383  21.724560  39.449790  67.378702  73.109297  80.350881  89.253120      100.0
k2/k1 = 8.0          0.0  11.542702  26.070922  47.342381  80.858940  84.291953  88.595273  93.813330      100.0
k2/k1 = 16.0         0.0  12.779433  28.864263  52.414830  89.522484  91.421372  93.792016  96.646648      100.0

FDM Heat Convection Results for x_bar = 0.5:
              2nd Order Heat Convection  4th Order Heat Convection
k2/k1 = 0.0625                -1.583037                  -1.603973
k2/k1 = 0.125                 -2.012353                  -2.038259
k2/k1 = 0.25                  -2.674530                  -2.701452
k2/k1 = 0.5                   -3.665089                  -3.690105
k2/k1 = 1.0                   -5.082312                  -5.103008
k2/k1 = 2.0                   -6.905215                  -6.914954
k2/k1 = 4.0                   -8.864016                  -8.851486
k2/k1 = 8.0                  -10.561319                 -10.519257
k2/k1 = 16.0                 -11.768665                 -11.699504

FDM Heat Convection Results for x_bar = 2/pi:
              2nd Order Heat Convection  4th Order Heat Convection
k2/k1 = 0.0625                -1.432463                  -1.253695
k2/k1 = 0.125                 -1.898803                  -1.763475
k2/k1 = 0.25                  -2.603336                  -2.502977
k2/k1 = 0.5                   -3.639854                  -3.564905
k2/k1 = 1.0                   -5.047653                  -4.985553
k2/k1 = 2.0                   -6.678556                  -6.611422
k2/k1 = 4.0                   -8.208090                  -8.119747
k2/k1 = 8.0                   -9.370579                  -9.256061
k2/k1 = 16.0                 -10.118258                  -9.982436

Section 2: FEM with Varying k_ratio
FEM p=1 Temperature Results x_bar = 0.5:
              x = 0.000  x = 0.125  x = 0.250  x = 0.375  x = 0.500  x = 0.625  x = 0.750  x = 0.875  x = 1.000
k2/k1 = 0.0625       0.0   0.008040   0.017369   0.029485   0.046331   0.431694   2.667176  16.333414      100.0
k2/k1 = 0.125        0.0   0.092421   0.199668   0.338947   0.532604   2.747608   9.300941  30.539969      100.0
k2/k1 = 0.25         0.0   0.461952   0.998014   1.694185   2.662150   8.197855  19.452994  44.279989      100.0
k2/k1 = 0.5          0.0   1.463251   3.161248   5.366397   8.432466  17.199749  31.637279  56.504681      100.0
k2/k1 = 1.0          0.0   3.408346   7.363486  12.499933  19.641717  29.852520  44.852497  67.048062      100.0
k2/k1 = 2.0          0.0   6.282365  13.572595  23.040247  36.204196  45.614623  58.635706  76.298138      100.0
k2/k1 = 4.0          0.0   9.556088  20.645239  35.046460  55.070107  62.227195  71.830962  84.259013      100.0
k2/k1 = 8.0          0.0  12.463187  26.925816  45.708094  71.823222  76.490410  82.656431  90.442107      100.0
k2/k1 = 16.0         0.0  14.553527  31.441845  53.374310  83.869497  86.594484  90.166500  94.620483      100.0

FEM p=2 Temperature Results x_bar = 0.5:
              x = 0.000  x = 0.125  x = 0.250  x = 0.375  x = 0.500  x = 0.625  x = 0.750  x = 0.875  x = 1.000
k2/k1 = 0.0625       0.0   0.012068   0.026047   0.044148   0.069238   0.867408   4.294089  20.733162      100.0
k2/k1 = 0.125        0.0   0.098218   0.211984   0.359304   0.563499   3.316456  10.653512  32.716155      100.0
```

```
k2/k1 = 0.25          0.0   0.465290   1.004231   1.702131   2.669460   8.632229  20.273710  45.252255      100.0
k2/k1 = 0.5           0.0   1.471126   3.175116   5.381693   8.440137  17.439069  32.030325  56.892999      100.0
k2/k1 = 1.0           0.0   3.440538   7.425677  12.586223  19.739039  30.016340  45.044917  67.203638      100.0
k2/k1 = 2.0           0.0   6.351749  13.708915  23.236059  36.441232  45.815587  58.792618  76.392764      100.0
k2/k1 = 4.0           0.0   9.653407  20.834849  35.314232  55.383492  62.464580  71.993637  84.344107      100.0
k2/k1 = 8.0           0.0  12.568039  27.125469  45.976580  72.105307  76.701038  82.797275  90.513280      100.0
k2/k1 = 16.0          0.0  14.652946  31.625303  53.603617  84.066829  86.741869  90.264686  94.669712      100.0


FEM p=1 Temperature Results x_bar = 2/pi:
              x = 0.000  x = 0.159  x = 0.318  x = 0.477  x = 0.637  x = 0.727  x = 0.818  x = 0.909  x = 1.000
k2/k1 = 0.0625        0.0   0.051867   0.117450   0.214096   0.367363   2.434575   8.623646  29.412987      100.0
k2/k1 = 0.125         0.0   0.275874   0.624708   1.138757   1.953971   7.451625  18.477330  43.210596      100.0
k2/k1 = 0.25          0.0   0.917965   2.078702   3.789190   6.501798  15.648464  30.261653  55.446231      100.0
k2/k1 = 0.5           0.0   2.241155   5.075024   9.251075  15.873740  27.039252  42.794017  65.812018      100.0
k2/k1 = 1.0           0.0   4.339198   9.825975  17.911409  30.733838  41.542855  55.828147  74.785094      100.0
k2/k1 = 2.0           0.0   6.933484  15.700652  28.620142  49.108745  57.744459  68.779447  82.672214      100.0
k2/k1 = 4.0           0.0   9.434441  21.363990  38.943632  66.822621  72.697960  80.078383  89.116689      100.0
k2/k1 = 8.0           0.0  11.358900  25.721867  46.887445  80.453254  83.990156  88.394995  93.713291      100.0
k2/k1 = 16.0          0.0  12.604001  28.541357  52.026991  89.272101  91.234400  93.667692  96.584539      100.0


FEM p=2 Temperature Results x_bar = 2/pi:
              x = 0.000  x = 0.159  x = 0.318  x = 0.477  x = 0.637  x = 0.727  x = 0.818  x = 0.909  x = 1.000
k2/k1 = 0.0625        0.0   0.056356   0.127290   0.231150   0.394802   3.000696  10.011857  31.721242      100.0
k2/k1 = 0.125         0.0   0.282083   0.637133   1.156990   1.976128   7.903319  19.338326  44.250198      100.0
k2/k1 = 0.25          0.0   0.936679   2.115650   3.841878   6.561891  15.925358  30.691359  55.869129      100.0
k2/k1 = 0.5           0.0   2.296689   5.187467   9.420092  16.089426  27.281478  43.038618  65.997532      100.0
k2/k1 = 1.0           0.0   4.451072  10.053508  18.256499  31.181922  41.889346  56.077632  74.925778      100.0
k2/k1 = 2.0           0.0   7.095212  16.025752  29.101694  49.705410  58.184789  69.073373  82.822018      100.0
k2/k1 = 4.0           0.0   9.618769  21.725639  39.452308  67.384157  73.113291  80.353498  89.254414      100.0
k2/k1 = 8.0           0.0  11.542780  26.071346  47.343822  80.862788  84.294805  88.597158  93.814267      100.0
k2/k1 = 16.0          0.0  12.779243  28.864110  52.415295  89.524814  91.423109  93.793169  96.647223      100.0


FEM Heat Convection Results for x_bar = 0.5:
              p=1 Heat Convection  p=2 Heat Convection
k2/k1 = 0.0625          -1.638864           -1.604308
k2/k1 = 0.125           -2.072823           -2.038640
k2/k1 = 0.25            -2.732243           -2.701699
k2/k1 = 0.5             -3.714639           -3.690240
k2/k1 = 1.0             -5.122610           -5.103060
k2/k1 = 2.0             -6.938675           -6.914893
k2/k1 = 4.0             -8.894031           -8.851258
k2/k1 = 8.0            -10.590389          -10.518839
k2/k1 = 16.0           -11.797909          -11.698927

FEM Heat Convection Results for x_bar = 2/pi:
              p=1 Heat Convection  p=2 Heat Convection
k2/k1 = 0.0625          -1.476318           -1.253926
k2/k1 = 0.125           -1.940923           -1.763642
k2/k1 = 0.25            -2.639437           -2.503068
k2/k1 = 0.5             -3.669207           -3.564914
k2/k1 = 1.0             -5.073389           -4.985398
k2/k1 = 2.0             -6.705769           -6.610952
k2/k1 = 4.0             -8.240802           -8.118855
k2/k1 = 8.0             -9.409744           -9.254767
k2/k1 = 16.0           -10.162539           -9.980847

Section 3 Temp: Convergence of FDM and FEM Temp with Varying k Ratio
Convergence of FDM Temperature at x_bar = 0.500 and k_ratio = 0.0625
          True T   FDM 2 T  FDM 2 Extrap T  FDM 2 % Error  FDM 2 Exact B  FDM 2 Extrap % Error  FDM 2 Extrap B   FDM 4 T  FDM 4 Extrap T  FDM 4 % Error  FDM 4 Exact B  FDM 4 Extrap % Error  FDM 4 Extrap B
N = 4    0.066937  0.400229            NaN       4.979197            NaN                   NaN             NaN  0.104810             NaN   5.658100e-01            NaN                   NaN             NaN
N = 8    0.066937  0.134628            NaN       1.011272      -2.299743                   NaN             NaN  0.071181             NaN   6.340026e-02      -3.157757                   NaN             NaN
N = 16   0.066937  0.082453       0.069697       0.183010      -2.125242            0.183010             NaN  0.067277        0.066765   7.677697e-03      -3.640568          7.677697e-03             NaN
N = 32   0.066937  0.070718       0.067312       0.050588      -2.036995            0.050588      -2.152561  0.066960        0.066932   4.195795e-04      -3.891466          4.195795e-04      -3.620631
N = 64   0.066937  0.067876       0.066968       0.013561      -2.009682            0.013561      -2.045906  0.066938        0.066937   2.326877e-05      -3.971390          2.326877e-05      -3.885859
N = 128  0.066937  0.067171       0.066939       0.003469      -2.002449            0.003469      -2.012080  0.066937        0.066937   1.395107e-06      -3.992773          1.395107e-06      -3.969945
N = 256  0.066937  0.066995       0.066937       0.000873      -2.000614            0.000873      -2.003060  0.066937        0.066937   8.620313e-08      -3.998578          8.620313e-08      -3.992525
N = 512  0.066937  0.066952       0.066937       0.000219      -2.000154            0.000219      -2.000768  0.066937        0.066937   5.560008e-09      -4.005552          5.560008e-09      -3.951767

Convergence of FEM Temperature at x_bar = 0.500 and k_ratio = 0.0625
          True T   FEM 2 T  FEM 2 Extrap T  FEM 2 % Error  FEM 2 Exact B  FEM 2 Extrap % Error  FEM 2 Extrap B   FEM 4 T  FEM 4 Extrap T  FEM 4 % Error  FEM 4 Exact B  FEM 4 Extrap % Error  FEM 4 Extrap B
N = 4    0.066937 -0.363773            NaN       6.434572            NaN                   NaN             NaN  0.125341             NaN   8.725219e-01            NaN                   NaN             NaN
N = 8    0.066937  0.046331            NaN       0.307843      -4.385579                   NaN             NaN  0.069238             NaN   3.437611e-02      -4.665713                   NaN             NaN
N = 16   0.066937  0.062673       0.063352       0.010706      -2.272928            0.010706             NaN  0.067069        0.066982   1.301621e-03      -4.117651          1.301621e-03             NaN
N = 32   0.066937  0.065888       0.066675       0.011806      -2.023150            0.011806      -2.345888  0.066945        0.066937   1.131235e-04      -4.025230          1.131235e-04      -4.123498
N = 64   0.066937  0.066671       0.066924       0.003772      -1.982240            0.003772      -2.036753  0.066937        0.066937   7.455960e-06      -4.006027          7.455960e-06      -4.026495
N = 128  0.066937  0.066870       0.066937       0.001005      -1.984061            0.001005      -1.981623  0.066937        0.066937   4.709251e-07      -4.001558          4.709251e-07      -4.006325
N = 256  0.066937  0.066920       0.066937       0.000254      -1.990328            0.000254      -1.981947  0.066937        0.066937   2.948335e-08      -4.001514          2.948335e-08      -4.002541
N = 512  0.066937  0.066933       0.066937       0.000064      -1.994748            0.000064      -1.988845  0.066937        0.066937   2.462299e-09      -4.018498          2.462299e-09      -3.613957

Convergence of FDM Temperature at x_bar = 0.500 and k_ratio = 0.125
          True T   FDM 2 T  FDM 2 Extrap T  FDM 2 % Error  FDM 2 Exact B  FDM 2 Extrap % Error  FDM 2 Extrap B   FDM 4 T  FDM 4 Extrap T  FDM 4 % Error  FDM 4 Exact B  FDM 4 Extrap % Error  FDM 4 Extrap B
N = 4    0.560194  1.321586            NaN       1.359157            NaN                   NaN             NaN  0.644433             NaN   1.503736e-01            NaN                   NaN             NaN
N = 8    0.560194  0.749289            NaN       0.337552      -2.009531                   NaN             NaN  0.568335             NaN   1.453274e-02      -3.371173                   NaN             NaN
N = 16   0.560194  0.607201       0.560273       0.083760      -2.008170            0.083760             NaN  0.560783        0.559951   1.485903e-03      -3.788394          1.485903e-03             NaN
N = 32   0.560194  0.571925       0.560274       0.020794      -2.002601            0.020794      -2.010016  0.560233        0.560189   7.737417e-05      -3.942014          7.737417e-05      -3.777069
N = 64   0.560194  0.563125       0.560201       0.005220      -2.000690            0.005220      -2.003237  0.560197        0.560194   4.471350e-06      -3.985145          4.471350e-06      -3.939060
N = 128  0.560194  0.562177       0.560195       0.001307      -2.000175            0.001307      -2.000862  0.560194        0.560194   2.731656e-07      -3.996264          2.731656e-07      -3.984413
N = 256  0.560194  0.560377       0.560194       0.000327      -2.000044            0.000327      -2.000219  0.560194        0.560194   1.695761e-08      -3.999156          1.695761e-08      -3.997252
N = 512  0.560194  0.560240       0.560194       0.000082      -2.000011            0.000082      -2.000055  0.560194        0.560194   1.457216e-09      -4.000608          1.457216e-09      -3.572239

Convergence of FEM Temperature at x_bar = 0.500 and k_ratio = 0.125
          True T   FEM 2 T  FEM 2 Extrap T  FEM 2 % Error  FEM 2 Exact B  FEM 2 Extrap % Error  FEM 2 Extrap B   FEM 4 T  FEM 4 Extrap T  FEM 4 % Error  FEM 4 Exact B  FEM 4 Extrap % Error  FEM 4 Extrap B
N = 4    0.560194  0.284007            NaN       0.493020            NaN                   NaN             NaN  0.622268             NaN   1.108077e-01            NaN                   NaN             NaN
N = 8    0.560194  0.532504            NaN       0.049252      -3.323401                   NaN             NaN  0.563499             NaN   5.898885e-03      -4.231472                   NaN             NaN
N = 16   0.560194  0.553343       0.555230       0.003400      -2.009667            0.003400             NaN  0.560394        0.560221   3.090697e-04      -4.046545          3.090697e-04             NaN
N = 32   0.560194  0.558340       0.559926       0.002833      -1.885500            0.002833      -2.053153  0.560207        0.560195   2.153037e-05      -4.010664          2.153037e-05      -4.048887
N = 64   0.560194  0.559704       0.560217       0.000915      -1.920215            0.000915      -1.872828  0.560195        0.560194   1.372921e-06      -4.002600          1.372921e-06      -4.011199
N = 128  0.560194  0.560068       0.560230       0.000236      -1.955459            0.000236      -1.907766  0.560194        0.560194   8.618886e-08      -4.000648          8.618886e-08      -4.002640
N = 256  0.560194  0.560162       0.560195       0.000059      -1.976687            0.000059      -1.948157  0.560194        0.560194   5.269850e-09      -4.000264          5.269850e-09      -4.031720
N = 512  0.560194  0.560186       0.560194       0.000015      -1.988098            0.000015      -1.972821  0.560194        0.560194   1.244402e-09      -4.007095          1.244402e-09      -1.615813

Convergence of FDM Temperature at x_bar = 0.500 and k_ratio = 0.25
          True T   FDM 2 T  FDM 2 Extrap T  FDM 2 % Error  FDM 2 Exact B  FDM 2 Extrap % Error  FDM 2 Extrap B   FDM 4 T  FDM 4 Extrap T  FDM 4 % Error  FDM 4 Exact B  FDM 4 Extrap % Error  FDM 4 Extrap B
N = 4    2.666666  3.862069            NaN       0.448276            NaN                   NaN             NaN  2.766445             NaN   3.741715e-02            NaN                   NaN             NaN
N = 8    2.666666  2.983333            NaN       0.118788      -1.916002                   NaN             NaN  2.674914             NaN   3.092841e-03      -3.596695                   NaN             NaN
N = 16   2.666666  2.747136       2.660209       0.032677      -1.976907            0.032677             NaN  2.667225        2.666520   2.643938e-04      -3.882410          2.643938e-04             NaN
N = 32   2.666666  2.686866       2.666231       0.007739      -1.994067            0.007739      -1.971109  2.666702        2.666664   1.434611e-05      -3.969306          1.434611e-05      -3.876288
N = 64   2.666666  2.671721       2.666638       0.001906      -1.998506            0.001906      -1.992582  2.666668        2.666666   8.567257e-07      -3.992241          8.567257e-07      -3.967753
N = 128  2.666666  2.667930       2.666664       0.000475      -1.999626            0.000475      -1.998132  2.666666        2.666666   5.289006e-08      -3.998054          5.289006e-08      -3.992024
N = 256  2.666666  2.666982       2.666666       0.000119      -1.999906            0.000119      -1.999532  2.666666        2.666666   2.955733e-09      -3.999619          2.955733e-09      -4.145736
N = 512  2.666666  2.666745       2.666666       0.000030      -1.999977            0.000030      -1.999883  2.666666        2.666666   1.072377e-09      -4.006796          1.072377e-09      -0.909712

Convergence of FEM Temperature at x_bar = 0.500 and k_ratio = 0.25
          True T   FEM 2 T  FEM 2 Extrap T  FEM 2 % Error  FEM 2 Exact B  FEM 2 Extrap % Error  FEM 2 Extrap B   FEM 4 T  FEM 4 Extrap T  FEM 4 % Error  FEM 4 Exact B  FEM 4 Extrap % Error  FEM 4 Extrap B
N = 4    2.666666  2.715719            NaN   1.839487e-02            NaN                   NaN             NaN  2.714187             NaN   1.782020e-02            NaN                   NaN             NaN
N = 8    2.666666  2.662150            NaN   1.693516e-03      -3.441210                   NaN             NaN  2.669460             NaN   1.047764e-03      -4.088127                   NaN             NaN
N = 16   2.666666  2.661848       2.661847   6.420123e-07       0.093302      6.420173e-07             NaN  2.666838        2.666675   6.121488e-05      -4.018859          6.121488e-05             NaN
N = 32   2.666666  2.664898       2.662123   1.042592e-03      -1.446493      1.042592e-03      3.337586  2.666677        2.666666   3.981784e-06      -4.004467          3.981784e-06      -4.019811
N = 64   2.666666  2.666147       2.667012   3.244563e-04      -1.767683      3.244563e-04      -1.288577  2.666667        2.666666   2.508414e-07      -4.001100          2.508414e-07      -4.004687
N = 128  2.666666  2.666526       2.666692   6.204964e-05      -1.891986      6.204964e-05      -1.718994  2.666666        2.666666   1.572281e-08      -4.000269          1.572281e-08      -3.999707
N = 256  2.666666  2.666630       2.666659   1.460619e-05      -1.947776      1.460619e-05      -1.871943  2.666666        2.666666   1.020082e-09      -4.000371          1.020082e-09      -3.948992
N = 512  2.666666  2.666657       2.666666   3.577514e-06      -1.974309      3.577514e-06      -1.938606  2.666666        2.666666   7.946340e-09      -4.026006          7.946340e-09       0.177873

Convergence of FDM Temperature at x_bar = 0.500 and k_ratio = 0.5
          True T   FDM 2 T  FDM 2 Extrap T  FDM 2 % Error  FDM 2 Exact B  FDM 2 Extrap % Error  FDM 2 Extrap B   FDM 4 T  FDM 4 Extrap T  FDM 4 % Error  FDM 4 Exact B  FDM 4 Extrap % Error  FDM 4 Extrap B
N = 4    8.438605  9.674633            NaN       0.146473            NaN                   NaN             NaN  8.503685             NaN   7.712231e-03            NaN                   NaN             NaN
N = 8    8.438605  8.767916            NaN       0.039024      -1.908190                   NaN             NaN  8.443410             NaN   5.694530e-04      -3.759500                   NaN             NaN
N = 16   8.438605  8.522370       8.431179       0.010816      -1.975035            0.010816             NaN  8.439919        8.438557   4.285047e-05      -3.935125          4.285047e-05             NaN
N = 32   8.438605  8.459639       8.438114       0.002551      -1.993614            0.002551      -1.968752  8.438625        8.438604   2.445375e-06      -3.983452          2.445375e-06      -3.931805
N = 64   8.438605  8.443869       8.438574       0.000628      -1.998394            0.000628      -1.992015  8.438606        8.438605   1.489725e-07      -3.995842          1.489725e-07      -3.982625
N = 128  8.438605  8.439921       8.438603       0.000156      -1.999598            0.000156      -1.997993  8.438605        8.438605   9.219334e-09      -3.998990          9.219334e-09      -4.000064
N = 256  8.438605  8.438934       8.438605       0.000039      -1.999899            0.000039      -1.999497  8.438605        8.438605   7.023049e-10      -4.001623          7.023049e-10      -3.735251
N = 512  8.438605  8.438687       8.438605       0.000010      -1.999975            0.000010      -1.999874  8.438605        8.438605   2.961137e-08      -4.057284          2.961137e-08      -0.026140

Convergence of FEM Temperature at x_bar = 0.500 and k_ratio = 0.5
          True T   FEM 2 T  FEM 2 Extrap T  FEM 2 % Error  FEM 2 Exact B  FEM 2 Extrap % Error  FEM 2 Extrap B   FEM 4 T  FEM 4 Extrap T  FEM 4 % Error  FEM 4 Exact B  FEM 4 Extrap % Error  FEM 4 Extrap B
N = 4    8.438605  8.652638            NaN       0.025364            NaN                   NaN             NaN  8.463770             NaN   2.982193e-03            NaN                   NaN             NaN
N = 8    8.438605  8.432466            NaN       0.000727      -5.123736                   NaN             NaN  8.440137             NaN   1.816114e-04      -4.037447                   NaN             NaN
N = 16   8.438605  8.429022       8.428967       0.000006       0.642537            0.000006             NaN  8.438700        8.438607   1.102988e-05      -4.008491          1.102988e-05             NaN
N = 32   8.438605  8.435145       8.431226       0.000465      -1.469764            0.000465      0.830074  8.438611        8.438605   7.006905e-07      -4.002060          7.006905e-07      -4.008915
N = 64   8.438605  8.437603       8.439252       0.000195      -1.788181            0.000195      -1.316727  8.438605        8.438605   4.395466e-08      -4.000510          4.395466e-08      -4.001907
N = 128  8.438605  8.438337       8.438650       0.000037      -1.903464            0.000037      -1.743720  8.438605        8.438605   2.756925e-09      -4.000105          2.756925e-09      -3.996608
N = 256  8.438605  8.438536       8.438609       0.000009      -1.953754            0.000009      -1.885546  8.438605        8.438605   9.335272e-09      -4.006893          9.335272e-09      -0.351898
N = 512  8.438605  8.438587       8.438605       0.000002      -1.977347            0.000002      -1.945635  8.438605        8.438605   3.033763e-08      -4.109874          3.033763e-08       0.007683

Convergence of FDM Temperature at x_bar = 0.500 and k_ratio = 1.0
          True T    FDM 2 T  FDM 2 Extrap T  FDM 2 % Error  FDM 2 Exact B  FDM 2 Extrap % Error  FDM 2 Extrap B    FDM 4 T  FDM 4 Extrap T  FDM 4 % Error  FDM 4 Exact B  FDM 4 Extrap % Error  FDM 4 Extrap B
N = 4    19.738549  20.447284            NaN       0.035906            NaN                   NaN             NaN  19.752762             NaN   7.200901e-04            NaN                   NaN             NaN
N = 8    19.738549  19.922720            NaN       0.009331      -1.944198                   NaN             NaN  19.739499             NaN   4.813553e-05      -3.903003                   NaN             NaN
N = 16   19.738549  19.785060       19.736081       0.002482      -1.985391            0.002482             NaN  19.738609        19.738545   3.241007e-06      -3.975074          3.241007e-06             NaN
N = 32   19.738549  19.750206       19.738390       0.000599      -1.996303            0.000599      -1.981722  19.738553        19.738549   1.949892e-07      -3.993725          1.949892e-07      -3.973832
N = 64   19.738549  19.741465       19.738539       0.000148      -1.999073            0.000148      -1.995378  19.738549        19.738549   1.196502e-08      -3.998427          1.196502e-08      -4.004838
N = 128  19.738549  19.739278       19.738548       0.000037      -1.999768            0.000037      -1.998841  19.738549        19.738549   7.604435e-10      -3.999475          7.604435e-10      -3.788746
N = 256  19.738549  19.738731       19.738549       0.000009      -1.999942            0.000009      -1.999710  19.738549        19.738549   1.678576e-08      -4.027237          1.678576e-08       0.061951
N = 512  19.738549  19.738594       19.738549       0.000002      -1.999986            0.000002      -1.999924  19.738549        19.738549   3.410828e-09      -3.453726          3.410828e-09      -0.017611
```

Convergence of FEM Temperature at x_bar = 0.500 and k_ratio = 1.0

	True T	FEM 2 T	FEM 2 Extrap T	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 T	FEM 4 Extrap T	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	19.738549	19.679634	NaN	0.002985	NaN	NaN	NaN	19.746583	NaN	4.070439e-04	NaN	NaN	NaN
N = 8	19.738549	19.641717	NaN	0.004906	0.716842	NaN	NaN	19.739039	NaN	2.483155e-05	-4.034938	NaN	NaN
N = 16	19.738549	19.703350	19.665192	0.001940	-1.459966	0.001940	NaN	19.738579	19.738549	1.511130e-06	-4.008826	1.511130e-06	NaN
N = 32	19.738549	19.728328	19.745349	0.000862	-1.784051	0.000862	-1.303049	19.738551	19.738549	9.574890e-08	-4.002212	9.574890e-08	-4.009400
N = 64	19.738549	19.735813	19.739016	0.000162	-1.901696	0.000162	-1.738548	19.738549	19.738549	6.080926e-09	-4.000546	6.080926e-09	-3.985562
N = 128	19.738549	19.737842	19.738597	0.000038	-1.952962	0.000038	-1.883408	19.738549	19.738549	5.916877e-10	-4.000085	5.916877e-10	-3.996266
N = 256	19.738549	19.738369	19.738554	0.000009	-1.976979	0.000009	-1.944692	19.738549	19.738548	1.464429e-08	-4.036310	1.464429e-08	-0.034358
N = 512	19.738549	19.738504	19.738549	0.000002	-1.988614	0.000002	-1.973033	19.738549	19.738557	4.197772e-07	-4.681374	4.197772e-07	0.000076

Convergence of FDM Temperature at x_bar = 0.500 and k_ratio = 2.0

	True T	FDM 2 T	FDM 2 Extrap T	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 T	FDM 4 Extrap T	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	36.441301	36.351342	NaN	2.468603e-03	NaN	NaN	NaN	36.420043	NaN	5.833573e-04	NaN	NaN	NaN
N = 8	36.441301	36.412116	NaN	8.008765e-04	-1.624043	NaN	NaN	36.439800	NaN	4.120741e-05	-3.823404	NaN	NaN
N = 16	36.441301	36.433533	36.445188	3.197816e-04	-1.909614	3.197816e-04	NaN	36.441312	36.441301	2.950879e-06	-3.954542	2.950879e-06	NaN
N = 32	36.441301	36.439329	36.441479	5.900323e-05	-1.977577	5.900323e-05	-1.885733	36.441295	36.441301	1.720071e-07	-3.988550	1.720071e-07	-3.952224
N = 64	36.441301	36.440806	36.441312	1.387073e-05	-1.994404	1.387073e-05	-1.971895	36.441301	36.441301	1.057241e-08	-3.997120	1.057241e-08	-3.986035
N = 128	36.441301	36.441177	36.441302	3.416958e-06	-1.998602	3.416958e-06	-1.993002	36.441301	36.441301	5.361641e-10	-3.998868	5.361641e-10	-4.115557
N = 256	36.441301	36.441270	36.441301	8.511403e-07	-1.999650	8.511403e-07	-1.998241	36.441301	36.441300	3.336626e-08	-3.967098	3.336626e-08	0.038464
N = 512	36.441301	36.441294	36.441301	2.127102e-07	-1.999937	2.127102e-07	-1.998956	36.441301	36.441308	1.884169e-07	-2.798008	1.884169e-07	-0.000275

Convergence of FEM Temperature at x_bar = 0.500 and k_ratio = 2.0

	True T	FEM 2 T	FEM 2 Extrap T	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 T	FEM 4 Extrap T	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	36.441301	35.832852	NaN	0.016697	NaN	NaN	NaN	36.440338	NaN	2.644585e-05	NaN	NaN	NaN
N = 8	36.441301	36.204196	NaN	0.006507	-1.359606	NaN	NaN	36.441232	NaN	1.912466e-06	-3.789535	NaN	NaN
N = 16	36.441301	36.370470	36.505289	0.001693	-1.743070	0.003693	-1.968566	36.441297	36.441302	1.406682e-07	-3.949079	1.406682e-07	-3.961178
N = 32	36.441301	36.422094	36.445339	0.000638	-1.882741	0.000638	-1.687447	36.441301	36.441301	8.056553e-09	-3.987367	8.056553e-09	-3.944881
N = 64	36.441301	36.436309	36.441710	0.000148	-1.943854	0.000148	-1.860649	36.441301	36.441301	4.410936e-10	-3.996880	4.410936e-10	-3.962608
N = 128	36.441301	36.440029	36.441348	0.000036	-1.972518	0.000036	-1.933921	36.441301	36.441302	2.726522e-08	-3.988805	2.726522e-08	-0.024043
N = 256	36.441301	36.441080	36.441307	0.000009	-1.986403	0.000009	-1.967801	36.441301	36.441285	4.388124e-07	-3.743232	4.388124e-07	0.000094
N = 512	36.441301	36.441221	36.441302	0.000002	-1.993237	0.000002	-1.984105	36.441301	36.441142	4.360262e-06	0.671980	4.360262e-06	0.000002

Convergence of FDM Temperature at x_bar = 0.500 and k_ratio = 4.0

	True T	FDM 2 T	FDM 2 Extrap T	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 T	FDM 4 Extrap T	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	55.383747	54.738450	NaN	1.165138e-02	NaN	NaN	NaN	55.350493	NaN	6.004303e-04	NaN	NaN	NaN
N = 8	55.383747	55.211305	NaN	3.113580e-03	-1.903854	NaN	NaN	55.381453	NaN	4.141831e-05	-3.857656	NaN	NaN
N = 16	55.383747	55.339880	55.387698	8.669335e-04	-1.974909	8.669335e-04	NaN	55.383600	55.383760	2.888108e-06	-3.963085	2.888108e-06	NaN
N = 32	55.383747	55.372432	55.384066	2.035705e-04	-1.993656	2.035705e-04	-1.968566	55.383737	55.383747	1.707474e-07	-3.949684	1.707474e-07	-3.961178
N = 64	55.383747	55.380990	55.383763	5.006916e-05	-1.998410	5.006916e-05	-1.992066	55.383746	55.383747	1.052346e-08	-3.997669	1.052346e-08	-3.989239
N = 128	55.383747	55.383057	55.383748	1.246579e-05	-1.999602	1.246579e-05	-1.998012	55.383747	55.383747	3.603694e-10	-3.998908	3.603694e-10	-4.817891
N = 256	55.383747	55.383574	55.383747	3.113229e-06	-1.999900	3.113229e-06	-1.999503	55.383747	55.383746	1.414303e-08	-4.010329	1.414303e-08	0.063983
N = 512	55.383747	55.383704	55.383747	7.781141e-07	-1.999979	7.781141e-07	-1.999865	55.383747	55.383739	1.297991e-07	-4.028543	1.297991e-07	0.000424

Convergence of FEM Temperature at x_bar = 0.500 and k_ratio = 4.0

	True T	FEM 2 T	FEM 2 Extrap T	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 T	FEM 4 Extrap T	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	55.383747	54.410234	NaN	0.017578	NaN	NaN	NaN	55.379824	NaN	7.082611e-05	NaN	NaN	NaN
N = 8	55.383747	55.070107	NaN	0.005663	-1.634093	NaN	NaN	55.383492	NaN	4.592963e-06	-3.946784	NaN	NaN
N = 16	55.383747	55.295589	55.412632	0.002112	-1.830954	0.002112	NaN	55.383731	55.383747	2.989977e-07	-3.986123	2.989977e-07	NaN
N = 32	55.383747	55.360424	55.386591	0.000472	-1.918370	0.000472	-1.798167	55.383746	55.383747	3.831005e-08	-3.996495	1.831004e-08	-3.985276
N = 64	55.383747	55.377752	55.384071	0.000114	-1.959856	0.000114	-1.903734	55.383747	55.383747	1.826820e-09	-3.999134	1.826820e-09	-3.367154
N = 128	55.383747	55.382227	55.383786	0.000028	-1.980090	0.000028	-1.952921	55.383747	55.383747	3.753217e-09	-3.997425	3.753217e-09	-0.360222
N = 256	55.383747	55.383364	55.383751	0.000007	-1.990085	0.000007	-1.976712	55.383747	55.383739	1.296623e-07	-3.951150	1.296623e-07	0.000740
N = 512	55.383747	55.383651	55.383747	0.000002	-1.995056	0.000002	-1.988421	55.383747	55.383907	2.899492e-06	-1.097392	2.899492e-06	-0.000003

Convergence of FDM Temperature at x_bar = 0.500 and k_ratio = 8.0

	True T	FDM 2 T	FDM 2 Extrap T	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 T	FDM 4 Extrap T	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	72.105546	71.363029	NaN	1.029764e-02	NaN	NaN	NaN	72.076518	NaN	4.025792e-04	NaN	NaN	NaN
N = 8	72.105546	71.911316	NaN	2.693691e-03	-1.934658	NaN	NaN	72.103555	NaN	2.761201e-05	-3.865905	NaN	NaN
N = 16	72.105546	72.056411	72.108626	7.241098e-04	-1.982937	7.241098e-04	NaN	72.105419	72.105557	1.913110e-06	-3.965081	1.913110e-06	NaN
N = 32	72.105546	72.093226	72.107366	1.735887e-04	-1.995685	1.735887e-04	-1.978646	72.105542	72.105547	1.134007e-07	-3.991177	1.134007e-07	-3.963367
N = 64	72.105546	72.102464	72.105559	4.292014e-05	-1.998918	4.292014e-05	-1.994605	72.105546	72.105546	6.975706e-09	-3.997776	6.975706e-09	-3.994622
N = 128	72.105546	72.104776	72.105547	1.070002e-05	-1.999729	1.070002e-05	-1.998648	72.105546	72.105547	1.299462e-09	-3.998574	1.299462e-09	-2.589904
N = 256	72.105546	72.105354	72.105547	2.673124e-06	-1.999933	2.673124e-06	-1.999662	72.105546	72.105545	1.857040e-08	-3.932799	1.857040e-08	0.031962
N = 512	72.105546	72.105498	72.105546	6.682154e-07	-1.999972	6.682154e-07	-1.999828	72.105546	72.105562	2.200364e-07	-1.539356	2.200364e-07	-0.000123

Convergence of FEM Temperature at x_bar = 0.500 and k_ratio = 8.0

	True T	FEM 2 T	FEM 2 Extrap T	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 T	FEM 4 Extrap T	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	72.105546	71.167084	NaN	0.013015	NaN	NaN	NaN	72.101833	NaN	5.150457e-05	NaN	NaN	NaN
N = 8	72.105546	71.823222	NaN	0.003915	-1.732945	NaN	NaN	72.105307	NaN	3.321129e-06	-3.954955	NaN	NaN
N = 16	72.105546	72.028173	72.121272	0.001291	-1.867448	0.001291	NaN	72.105531	72.105547	2.148524e-07	-3.988195	2.148524e-07	NaN
N = 32	72.105546	72.085293	72.107363	0.000306	-1.933672	0.000306	-1.843217	72.105546	72.105546	1.314747e-08	-3.997014	1.314747e-08	-3.992830
N = 64	72.105546	72.100365	72.105768	0.000075	-1.966785	0.000075	-1.922112	72.105546	72.105546	9.292971e-10	-3.999105	9.292971e-10	-3.830052
N = 128	72.105546	72.104236	72.105574	0.000019	-1.983374	0.000019	-1.961125	72.105546	72.105548	2.624584e-08	-3.997261	2.624584e-08	-0.041629
N = 256	72.105546	72.105217	72.105550	0.000005	-1.991682	0.000005	-1.980572	72.105546	72.105550	5.247377e-08	-4.811127	5.247377e-08	-0.001360
N = 512	72.105546	72.105464	72.105547	0.000001	-1.995831	0.000001	-1.990264	72.105546	72.105640	1.292537e-06	1.420703	1.292537e-06	0.000003

Convergence of FDM Temperature at x_bar = 0.500 and k_ratio = 16.0

	True T	FDM 2 T	FDM 2 Extrap T	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 T	FDM 4 Extrap T	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	84.066993	83.499338	NaN	6.745276e-03	NaN	NaN	NaN	84.047391	NaN	2.331775e-04	NaN	NaN	NaN
N = 8	84.066993	83.920263	NaN	1.745393e-03	-1.950326	NaN	NaN	84.065651	NaN	1.596694e-05	-3.868269	NaN	NaN
N = 16	84.066993	84.029979	84.068733	4.609836e-04	-1.986999	4.609836e-04	NaN	84.066907	84.067000	1.104246e-06	-3.965621	1.104246e-06	NaN
N = 32	84.066993	84.057718	84.067105	1.116594e-04	-1.996710	1.116594e-04	-1.983738	84.066988	84.066993	6.557651e-08	-3.991309	6.557651e-08	-3.963928
N = 64	84.066993	84.064673	84.067180	2.768112e-05	-1.999175	2.768112e-05	-1.995887	84.066993	84.066993	3.988095e-09	-3.997789	3.988095e-09	-4.009368
N = 128	84.066993	84.066413	84.066994	6.905557e-06	-1.999794	6.905557e-06	-1.998968	84.066993	84.066993	8.955373e-10	-4.000157	8.955373e-10	-2.381555
N = 256	84.066993	84.066848	84.066993	1.725477e-06	-1.999949	1.725477e-06	-1.999734	84.066993	84.066996	3.310265e-08	-3.987292	3.310265e-08	-0.010228
N = 512	84.066993	84.066957	84.066993	4.313023e-07	-1.999980	4.313023e-07	-1.999948	84.066993	84.067022	3.427462e-07	-1.022110	3.427462e-07	-0.000034

Convergence of FEM Temperature at x_bar = 0.500 and k_ratio = 16.0

	True T	FEM 2 T	FEM 2 Extrap T	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 T	FEM 4 Extrap T	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	84.066993	83.389649	NaN	8.057211e-03	NaN	NaN	NaN	84.064444	NaN	3.031942e-05	NaN	NaN	NaN
N = 8	84.066993	83.969497	NaN	2.349269e-03	-1.778069	NaN	NaN	84.066229	NaN	1.852881e-06	-3.956566	NaN	NaN
N = 16	84.066993	84.013509	84.075262	7.345061e-04	-1.884628	7.345061e-04	NaN	84.066983	84.066993	1.261802e-07	-3.988602	1.261802e-07	NaN
N = 32	84.066993	84.053063	84.068040	1.781610e-04	-1.940876	1.781610e-04	-1.864285	84.066993	84.066993	7.651180e-09	-3.997117	7.651180e-09	-4.006481
N = 64	84.066993	84.063437	84.067126	4.387892e-05	-1.970029	4.387892e-05	-1.930747	84.066993	84.066993	1.028087e-09	-3.999169	1.028087e-09	-2.591014
N = 128	84.066993	84.065595	84.066995	1.088523e-05	-1.985006	1.088523e-05	-1.964966	84.066993	84.066993	1.353058e-10	-3.999538	1.353058e-10	0.019564
N = 256	84.066993	84.066767	84.066995	2.710530e-06	-1.992425	2.710530e-06	-1.982372	84.066993	84.066957	4.318022e-07	-4.806789	4.318022e-07	0.000097
N = 512	84.066993	84.066937	84.066993	6.762546e-07	-1.996214	6.762546e-07	-1.991181	84.066993	84.066953	4.734119e-07	1.292403	4.734119e-07	-0.000005

Convergence of FDM Temperature at x_bar = 0.637 and k_ratio = 0.0625

	True T	FDM 2 T	FDM 2 Extrap T	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 T	FDM 4 Extrap T	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	0.392138	0.964920	NaN	1.460667	NaN	NaN	NaN	0.458151	NaN	1.683420e-01	NaN	NaN	NaN
N = 8	0.392138	0.534322	NaN	0.362587	-2.010228	NaN	NaN	0.398641	NaN	1.658442e-02	-3.343494	NaN	NaN
N = 16	0.392138	0.427450	0.392169	0.089965	-2.009513	0.089965	NaN	0.392612	0.391933	1.733889e-03	-3.775895	1.733889e-03	NaN
N = 32	0.392138	0.400947	0.392207	0.022284	-2.003074	0.022284	-2.011646	0.392169	0.392133	8.993455e-05	-3.938252	8.993455e-05	-3.763853
N = 64	0.392138	0.394339	0.392144	0.005597	-2.000819	0.005597	-2.003825	0.392140	0.392138	5.175510e-06	-3.984440	5.175510e-06	-3.935083
N = 128	0.392138	0.392688	0.392138	0.001402	-2.000209	0.001402	-2.001022	0.392138	0.392138	3.157303e-07	-4.000641	3.157303e-07	-3.983349
N = 256	0.392138	0.392275	0.392138	0.000351	-2.000055	0.000351	-2.000260	0.392138	0.392138	1.959448e-08	-4.075163	1.959448e-08	-3.996810
N = 512	0.392138	0.392172	0.392138	0.000088	-2.000026	0.000088	-2.000065	0.392138	0.392138	1.347800e-09	-6.903970	1.347800e-09	-3.868545

Convergence of FEM Temperature at x_bar = 0.637 and k_ratio = 0.0625

	True T	FEM 2 T	FEM 2 Extrap T	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 T	FEM 4 Extrap T	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	0.392138	0.149088	NaN	0.619806	NaN	NaN	NaN	0.442756	NaN	1.290824e-01	NaN	NaN	NaN
N = 8	0.392138	0.367363	NaN	0.063178	-3.294328	NaN	NaN	0.394802	NaN	6.793787e-03	-4.247932	NaN	NaN
N = 16	0.392138	0.386079	0.387834	0.004526	-2.031788	0.004526	NaN	0.392299	0.392161	3.515340e-04	-4.048947	3.515340e-04	NaN
N = 32	0.392138	0.390525	0.391911	0.003535	-1.909892	0.003535	-2.073563	0.392148	0.392138	2.470996e-05	-4.011164	2.470996e-05	-4.051410
N = 64	0.392138	0.391716	0.392151	0.001110	-1.934512	0.001110	-1.901067	0.392138	0.392138	1.577555e-06	-4.003611	1.577555e-06	-4.011665
N = 128	0.392138	0.392029	0.392142	0.000286	-1.962964	0.000286	-1.924564	0.392138	0.392138	9.904100e-08	-4.015325	9.904100e-08	-4.002930
N = 256	0.392138	0.392110	0.392138	0.000071	-1.980497	0.000071	-1.956965	0.392138	0.392138	6.265574e-09	-4.257532	6.265574e-09	-3.985877
N = 512	0.392138	0.392131	0.392138	0.000018	-1.989961	0.000018	-1.977299	0.392138	0.392138	9.583398e-10	-2.911770	9.583398e-10	-2.340668

Convergence of FDM Temperature at x_bar = 0.637 and k_ratio = 0.125

	True T	FDM 2 T	FDM 2 Extrap T	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 T	FDM 4 Extrap T	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	1.973771	2.884832	NaN	0.461584	NaN	NaN	NaN	2.054428	NaN	4.086461e-02	NaN	NaN	NaN
N = 8	1.973771	2.217011	NaN	0.123236	-1.905166	NaN	NaN	1.980577	NaN	3.448250e-03	-3.566916	NaN	NaN
N = 16	1.973771	1.993705	1.968139	0.010330	-1.973570	0.034330	NaN	1.974235	1.973640	3.018150e-04	-3.872719	3.018150e-04	NaN
N = 32	1.973771	1.989328	1.973387	0.008078	-1.993183	0.008078	-1.966930	1.973800	1.973768	1.622337e-05	-3.966691	1.622337e-05	-3.866069
N = 64	1.973771	1.977665	1.973746	0.001985	-1.998282	0.001985	-1.991477	1.973773	1.973771	9.652899e-07	-3.991576	9.652899e-07	-3.965005
N = 128	1.973771	1.974744	1.973769	0.000494	-1.999570	0.000494	-1.997852	1.973771	1.973771	5.954558e-08	-3.997927	5.954558e-08	-3.990989
N = 256	1.973771	1.974014	1.973771	0.000123	-1.999892	0.000123	-1.999462	1.973771	1.973771	3.607388e-09	-4.003077	3.607388e-09	-4.035270
N = 512	1.973771	1.973832	1.973771	0.000031	-1.999973	0.000031	-1.999865	1.973771	1.973771	5.886205e-10	-3.994490	5.886205e-10	-2.785349

Convergence of FEM Temperature at x_bar = 0.637 and k_ratio = 0.125

	True T	FEM 2 T	FEM 2 Extrap T	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 T	FEM 4 Extrap T	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	1.973771	1.950437	NaN	0.011822	NaN	NaN	NaN	2.014040	NaN	2.040236e-02	NaN	NaN	NaN
N = 8	1.973771	1.953971	NaN	0.010032	-0.236949	NaN	NaN	1.976128	NaN	1.194381e-03	-4.094401	NaN	NaN
N = 16	1.973771	1.973537	1.948881	0.000546	-1.265896	0.008546	NaN	1.973916	1.973779	5.944648e-05	-4.019974	6.944648e-05	NaN
N = 32	1.973771	1.971213	1.976683	0.002767	-1.686822	0.002767	-1.026937	1.973780	1.973771	4.531587e-06	-4.004707	4.531587e-06	-4.020982
N = 64	1.973771	1.973063	1.973958	0.000453	-1.854433	0.000453	-1.617231	1.973771	1.973771	2.856214e-07	-4.001166	2.856214e-07	-4.004940
N = 128	1.973771	1.973585	1.973790	0.000104	-1.929744	0.000104	-1.826653	1.973771	1.973771	1.783865e-08	-4.000415	1.783865e-08	-4.004762
N = 256	1.973771	1.973723	1.973773	0.000025	-1.965480	0.000025	-1.917237	1.973771	1.973771	9.406020e-10	-4.002193	9.406020e-10	-4.235639
N = 512	1.973771	1.973759	1.973771	0.000006	-1.982889	0.000006	-1.959536	1.973771	1.973771	2.906984e-09	-4.013522	2.906984e-09	0.644742

Convergence of FDM Temperature at x_bar = 0.637 and k_ratio = 0.25

	True T	FDM 2 T	FDM 2 Extrap T	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 T	FDM 4 Extrap T	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	6.560619	7.452341	NaN	0.135920	NaN	NaN	NaN	6.606093	NaN	6.931221e-03	NaN	NaN	NaN
N = 8	6.560619	6.799021	NaN	0.036338	-1.903201	NaN	NaN	6.564006	NaN	5.161932e-04	-3.747126	NaN	NaN
N = 16	6.560619	6.621332	6.554951	0.010127	-1.973315	0.010127	NaN	6.560841	6.560584	3.921835e-05	-3.931271	3.921835e-05	NaN
N = 32	6.560619	6.575870	6.560239	0.002383	-1.993146	0.002383	-1.966601	6.560633	6.560619	2.229184e-06	-3.982433	2.229184e-06	-3.927750
N = 64	6.560619	6.564437	6.560595	0.000586	-1.998275	0.000586	-1.991430	6.560620	6.560619	1.356016e-07	-3.995583	1.356016e-07	-3.981561
N = 128	6.560619	6.561574	6.560618	0.000146	-1.999568	0.000146	-1.997843	6.560619	6.560619	8.425880e-09	-3.998939	8.425880e-09	-3.993782
N = 256	6.560619	6.560858	6.560619	0.000036	-1.999892	0.000036	-1.999460	6.560619	6.560619	1.339557e-09	-4.001360	1.339557e-09	-2.781327
N = 512	6.560619	6.560679	6.560619	0.000009	-1.999973	0.000009	-1.999865	6.560619	6.560619	1.112903e-08	-3.955342	1.112903e-08	-0.062203

Convergence of FEM Temperature at x_bar = 0.637 and k_ratio = 0.25

	True T	FEM 2 T	FEM 2 Extrap T	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 T	FEM 4 Extrap T	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	6.560619	6.572238	NaN	0.001771	NaN	NaN	NaN	6.581686	NaN	3.211104e-03	NaN	NaN	NaN
N = 8	6.560619	6.501798	NaN	0.008966	2.339890	NaN	NaN	6.561891	NaN	1.938194e-04	-4.050285	NaN	NaN
N = 16	6.560619	6.537168	6.525344	0.001812	-1.326678	0.001812	NaN	6.560698	6.560622	1.165695e-05	-4.011875	1.165695e-05	NaN
N = 32	6.560619	6.553889	6.567820	0.002167	-1.738076	0.002167	-1.106969	6.560624	6.560619	7.441118e-07	-4.002917	7.441118e-07	-4.012469
N = 64	6.560619	6.558712	6.561034	0.000354	-1.881813	0.000354	-1.680666	6.560620	6.560619	4.675026e-08	-4.000727	4.675026e-08	-4.002539

N = 128	6.560619	6.560124	6.560661	0.000082	-1.943682	0.000082	-1.859435	6.560619	6.560619	2.986564e-09	-4.000190	2.986564e-09	-3.972231
N = 256	6.560619	6.560493	6.560624	0.000020	-1.972494	0.000020	-1.933696	6.560619	6.560619	5.124541e-09	-4.004226	5.124541e-09	-0.618512
N = 512	6.560619	6.560588	6.560620	0.000005	-1.986406	0.000005	-1.967768	6.560619	6.560619	5.223584e-08	-3.907740	5.223584e-08	-0.004692

Convergence of FDM Temperature at x_bar = 0.637 and k_ratio = 0.5

	True T	FDM 2 T	FDM 2 Extrap T	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 T	FDM 4 Extrap T	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	16.08921	16.377482	NaN	0.017917	NaN	NaN	NaN	16.072588	NaN	1.033089e-03	NaN	NaN	NaN
N = 8	16.08921	16.157318	NaN	0.004233	-2.081543	NaN	NaN	16.087851	NaN	8.444136e-05	-3.612871	NaN	NaN
N = 16	16.08921	16.105900	16.090233	0.000974	-2.028836	0.000974	NaN	16.089119	16.089234	7.136454e-06	-3.901042	7.136454e-06	NaN
N = 32	16.08921	16.093360	16.089315	0.000251	-2.007904	0.000251	-2.035696	16.089204	16.089210	3.811676e-07	-3.975106	3.811676e-07	-3.895879
N = 64	16.08921	16.090246	16.089218	0.000064	-2.002023	0.000064	-2.009856	16.089210	16.089210	2.290596e-08	-3.993769	2.290596e-08	-3.973248
N = 128	16.08921	16.089469	16.089211	0.000016	-2.000509	0.000016	-2.002527	16.089210	16.089210	2.318551e-09	-3.998132	2.318551e-09	-3.339367
N = 256	16.08921	16.089275	16.089210	0.000004	-2.000127	0.000004	-2.000635	16.089210	16.089210	2.682228e-09	-4.007949	2.682228e-09	0.981833
N = 512	16.08921	16.089226	16.089210	0.000001	-2.000027	0.000001	-2.000139	16.089210	16.089212	1.419220e-07	-3.851093	1.419220e-07	-0.000830

Convergence of FEM Temperature at x_bar = 0.637 and k_ratio = 0.5

	True T	FEM 2 T	FEM 2 Extrap T	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 T	FEM 4 Extrap T	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	16.08921	15.648397	NaN	0.027398	NaN	NaN	NaN	16.093157	NaN	2.453157e-04	NaN	NaN	NaN
N = 8	16.08921	15.873740	NaN	0.013392	-1.032680	NaN	NaN	16.089426	NaN	1.343464e-05	-4.190611	NaN	NaN
N = 16	16.08921	16.020446	16.294141	0.016797	-1.647759	0.016797	NaN	16.089223	16.089211	7.271523e-07	-4.055423	7.271523e-07	NaN
N = 32	16.08921	16.070057	16.095406	0.001575	-1.844072	0.001575	-1.564197	16.089211	16.089210	4.840807e-08	-4.014442	4.840807e-08	-4.058229
N = 64	16.08921	16.084171	16.089784	0.000349	-1.926462	0.000349	-1.813482	16.089210	16.089210	2.737377e-09	-4.003653	2.737377e-09	-4.180015
N = 128	16.08921	16.087919	16.089273	0.000084	-1.964289	0.000084	-1.913195	16.089210	16.089210	3.562759e-09	-4.000903	3.562759e-09	2.475812
N = 256	16.08921	16.088883	16.089218	0.000021	-1.982405	0.000021	-1.958100	16.089210	16.089210	3.995362e-09	-3.978635	3.995362e-09	-0.064383
N = 512	16.08921	16.089128	16.089211	0.000005	-1.991265	0.000005	-1.979413	16.089210	16.089220	6.038491e-07	-3.906710	6.038491e-07	0.000031

Convergence of FDM Temperature at x_bar = 0.637 and k_ratio = 1.0

	True T	FDM 2 T	FDM 2 Extrap T	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 T	FDM 4 Extrap T	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	31.182398	30.514477	NaN	0.021420	NaN	NaN	NaN	31.111549	NaN	2.272105e-03	NaN	NaN	NaN
N = 8	31.182398	30.989359	NaN	0.006191	-1.790787	NaN	NaN	31.177173	NaN	1.675687e-04	-3.761205	NaN	NaN
N = 16	31.182398	31.132234	31.193717	0.001971	-1.944144	0.001971	NaN	31.182057	31.182450	1.259443e-05	-3.937174	1.259443e-05	NaN
N = 32	31.182398	31.169733	31.183078	0.000428	-1.985785	0.000428	-1.929804	31.182377	31.182399	7.174384e-07	-3.984083	7.174384e-07	-3.933953
N = 64	31.182398	31.179224	31.182440	0.000103	-1.996430	0.000103	-1.982208	31.182397	31.182398	4.374637e-08	-3.996006	4.374637e-08	-3.982923
N = 128	31.182398	31.181604	31.182401	0.000026	-1.999106	0.000026	-1.995536	31.182398	31.182398	2.606778e-09	-3.998900	2.606778e-09	-4.051425
N = 256	31.182398	31.182200	31.182398	0.000006	-1.999777	0.000006	-1.998883	31.182398	31.182398	3.856539e-09	-3.998694	3.856539e-09	1.550952
N = 512	31.182398	31.182349	31.182398	0.000002	-1.999941	0.000002	-1.999717	31.182398	31.182398	1.478152e-08	-3.697036	1.478152e-08	0.015351

Convergence of FEM Temperature at x_bar = 0.637 and k_ratio = 1.0

	True T	FEM 2 T	FEM 2 Extrap T	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 T	FEM 4 Extrap T	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	31.182398	29.932292	NaN	0.040090	NaN	NaN	NaN	31.175417	NaN	2.238893e-04	NaN	NaN	NaN
N = 8	31.182398	30.733838	NaN	0.014385	-1.478678	NaN	NaN	31.181922	NaN	1.527575e-05	-3.873470	NaN	NaN
N = 16	31.182398	31.050756	31.258002	0.006630	-1.768681	0.006630	NaN	31.182368	31.182401	1.052223e-06	-3.967146	1.052223e-06	NaN
N = 32	31.182398	31.146901	31.188771	0.001342	-1.890838	0.001342	-1.720830	31.182396	31.182398	6.260357e-08	-3.991707	6.260357e-08	-3.965417
N = 64	31.182398	31.173192	31.183087	0.000317	-1.947032	0.000317	-1.870631	31.182398	31.182398	3.987232e-09	-3.997921	3.987232e-09	-3.948103
N = 128	31.182398	31.180055	31.182479	0.000078	-1.973922	0.000078	-1.937733	31.182398	31.182398	3.060232e-09	-3.998459	3.060232e-09	2.496761
N = 256	31.182398	31.181807	31.182408	0.000019	-1.987063	0.000019	-1.969462	31.182398	31.182398	1.082074e-08	-4.017430	1.082074e-08	0.030395
N = 512	31.182398	31.182250	31.182400	0.000005	-1.993556	0.000005	-1.984879	31.182398	31.182407	2.634205e-07	-4.040315	2.634205e-07	-0.000086

Convergence of FDM Temperature at x_bar = 0.637 and k_ratio = 2.0

	True T	FDM 2 T	FDM 2 Extrap T	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 T	FDM 4 Extrap T	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	49.706164	48.318359	NaN	0.027920	NaN	NaN	NaN	49.611795	NaN	1.898533e-03	NaN	NaN	NaN
N = 8	49.706164	49.325368	NaN	0.007661	-1.865716	NaN	NaN	49.699312	NaN	1.378513e-04	-3.783700	NaN	NaN
N = 16	49.706164	49.608556	49.719351	0.002228	-1.963950	0.002228	NaN	49.705718	49.706224	1.017998e-05	-3.942655	1.017998e-05	NaN
N = 32	49.706164	49.681606	49.707000	0.000511	-1.990813	0.000511	-1.954806	49.706136	49.706165	5.854912e-07	-3.985442	5.854912e-07	-3.939727
N = 64	49.706164	49.700015	49.706216	0.000125	-1.997692	0.000125	-1.988508	49.706162	49.706164	3.576368e-08	-3.996343	3.576368e-08	-3.984994
N = 128	49.706164	49.704626	49.706167	0.000031	-1.999422	0.000031	-1.997114	49.706164	49.706164	1.928803e-09	-3.999244	1.928803e-09	-4.188806
N = 256	49.706164	49.705779	49.706164	0.000008	-1.999856	0.000008	-1.999278	49.706164	49.706164	3.469626e-09	-3.992717	3.469626e-09	-0.677194
N = 512	49.706164	49.706068	49.706164	0.000002	-1.999965	0.000002	-1.999824	49.706164	49.706165	1.676751e-08	-4.252497	1.676751e-08	-0.011313

Convergence of FEM Temperature at x_bar = 0.637 and k_ratio = 2.0

	True T	FEM 2 T	FEM 2 Extrap T	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 T	FEM 4 Extrap T	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	49.706164	47.861343	NaN	0.037115	NaN	NaN	NaN	49.694823	NaN	2.281547e-04	NaN	NaN	NaN
N = 8	49.706164	49.108745	NaN	0.012019	-1.626666	NaN	NaN	49.705410	NaN	1.516824e-05	-3.910886	NaN	NaN
N = 16	49.706164	49.536981	49.760850	0.004499	-1.820156	0.004499	NaN	49.706116	49.706166	1.015037e-06	-3.976421	1.015037e-06	NaN
N = 32	49.706164	49.661198	49.711952	0.001021	-1.911699	0.001021	-1.785533	49.706161	49.706164	6.131217e-08	-3.994020	6.131217e-08	-3.975493
N = 64	49.706164	49.694576	49.706841	0.000247	-1.956261	0.000247	-1.895902	49.706164	49.706164	3.568787e-09	-3.998501	3.568787e-09	-4.077754
N = 128	49.706164	49.703223	49.706246	0.000061	-1.978235	0.000061	-1.948711	49.706164	49.706164	1.470481e-09	-4.000422	1.470481e-09	-1.770777
N = 256	49.706164	49.705423	49.706174	0.000015	-1.989144	0.000015	-1.974544	49.706164	49.706166	4.850477e-08	-3.922125	4.850477e-08	-0.006553
N = 512	49.706164	49.705978	49.706165	0.000004	-1.994578	0.000004	-1.987318	49.706164	49.706153	2.272247e-07	-5.434926	2.272247e-07	0.000101

Convergence of FDM Temperature at x_bar = 0.637 and k_ratio = 4.0

	True T	FDM 2 T	FDM 2 Extrap T	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 T	FDM 4 Extrap T	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	67.384857	65.895105	NaN	0.022108	NaN	NaN	NaN	67.299703	NaN	1.263688e-03	NaN	NaN	NaN
N = 8	67.384857	66.985431	NaN	0.005928	-1.899072	NaN	NaN	67.378702	NaN	9.133429e-05	-3.790340	NaN	NaN
N = 16	67.384857	67.283107	67.394899	0.001659	-1.972907	0.001659	NaN	67.384457	67.384909	6.709802e-06	-3.944130	6.709802e-06	NaN
N = 32	67.384857	67.359297	67.385506	0.000389	-1.993096	0.000389	-1.966071	67.384831	67.384858	3.870843e-07	-3.985796	3.870843e-07	-3.941284
N = 64	67.384857	67.378459	67.384898	0.000096	-1.998265	0.000096	-1.991365	67.384855	67.384857	2.370051e-08	-3.996432	2.370051e-08	-3.982999
N = 128	67.384857	67.383257	67.384859	0.000024	-1.999566	0.000024	-1.997832	67.384857	67.384857	1.671876e-09	-3.999119	1.671876e-09	-3.823551
N = 256	67.384857	67.384457	67.384857	0.000006	-1.999891	0.000006	-1.999458	67.384857	67.384856	3.788890e-09	-3.982202	3.788890e-09	0.650219
N = 512	67.384857	67.384757	67.384857	0.000001	-1.999966	0.000001	-1.999863	67.384857	67.384869	1.874138e-07	-4.196962	1.874138e-07	-0.000676

Convergence of FEM Temperature at x_bar = 0.637 and k_ratio = 4.0

	True T	FEM 2 T	FEM 2 Extrap T	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 T	FEM 4 Extrap T	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	67.384857	65.562359	NaN	0.027046	NaN	NaN	NaN	67.374295	NaN	1.567351e-04	NaN	NaN	NaN
N = 8	67.384857	66.822621	NaN	0.008344	-1.696669	NaN	NaN	67.384157	NaN	1.037839e-05	-3.916674	NaN	NaN
N = 16	67.384857	67.228445	67.421196	0.002859	-1.845831	0.002859	NaN	67.384812	67.384859	6.914151e-07	-3.977871	6.914151e-07	NaN
N = 32	67.384857	67.343581	67.389183	0.000677	-1.921976	0.000677	-1.817524	67.384854	67.384857	4.190551e-08	-3.994383	4.190551e-08	-3.975621
N = 64	67.384857	67.374253	67.385391	0.000165	-1.960712	0.000165	-1.908339	67.384857	67.384857	2.046963e-09	-3.998556	2.046963e-09	-4.316630
N = 128	67.384857	67.382169	67.384923	0.000041	-1.980281	0.000041	-1.954008	67.384857	67.384857	2.172310e-09	-4.000123	2.172310e-09	3.109563
N = 256	67.384857	67.384180	67.384865	0.000010	-1.990121	0.000010	-1.976956	67.384857	67.384855	3.177752e-08	-4.072990	3.177752e-08	0.006917
N = 512	67.384857	67.384687	67.384858	0.000003	-1.995056	0.000003	-1.988465	67.384857	67.384981	1.850506e-06	-5.906415	1.850506e-06	-0.000007

Convergence of FDM Temperature at x_bar = 0.637 and k_ratio = 8.0

	True T	FDM 2 T	FDM 2 Extrap T	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 T	FDM 4 Extrap T	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	80.863285	79.714831	NaN	1.420242e-02	NaN	NaN	NaN	80.803072	NaN	7.446247e-04	NaN	NaN	NaN
N = 8	80.863285	80.559732	NaN	3.753903e-03	-1.919673	NaN	NaN	80.858940	NaN	5.373535e-05	-3.792570	NaN	NaN
N = 16	80.863285	80.786251	80.869227	1.026052e-03	-1.978385	1.026052e-03	NaN	80.863003	80.863321	3.940703e-06	-3.944551	3.940703e-06	NaN
N = 32	80.863285	80.843953	80.863675	2.438962e-04	-1.994488	2.438962e-04	-1.972949	80.863267	80.863285	2.276011e-07	-3.985891	2.276011e-07	-3.941695
N = 64	80.863285	80.858447	80.863310	6.013105e-05	-1.998615	6.013105e-05	-1.993108	80.863284	80.863285	1.389183e-08	-3.996466	1.389183e-08	-3.987572
N = 128	80.863285	80.862075	80.863286	1.497913e-05	-1.999653	1.497913e-05	-1.998269	80.863285	80.863285	6.440270e-10	-3.998829	6.440270e-10	-4.254628
N = 256	80.863285	80.862982	80.863285	3.741411e-06	-1.999913	3.741411e-06	-1.999568	80.863285	80.863284	1.479848e-08	-4.006559	1.479848e-08	0.081162
N = 512	80.863285	80.863209	80.863285	9.351445e-07	-1.999979	9.351445e-07	-1.999890	80.863285	80.863293	9.851203e-08	-4.963517	9.851203e-08	-0.000812

Convergence of FEM Temperature at x_bar = 0.637 and k_ratio = 8.0

	True T	FEM 2 T	FEM 2 Extrap T	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 T	FEM 4 Extrap T	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	80.863285	79.498914	NaN	0.016873	NaN	NaN	NaN	80.855768	NaN	9.295800e-05	NaN	NaN	NaN
N = 8	80.863285	80.453254	NaN	0.005071	-1.734431	NaN	NaN	80.862788	NaN	6.149609e-06	-3.918013	NaN	NaN
N = 16	80.863285	80.750296	80.884534	0.001660	-1.859558	0.001660	NaN	80.863253	80.863286	4.092696e-07	-3.978196	4.092696e-07	NaN
N = 32	80.863285	80.833578	80.866025	0.000401	-1.927308	0.000401	-1.834600	80.863283	80.863285	2.482608e-08	-3.994463	2.482608e-08	-3.975571
N = 64	80.863285	80.855665	80.863637	0.000099	-1.962954	0.000099	-1.914803	80.863285	80.863285	1.084340e-09	-3.998642	1.084340e-09	-4.470206
N = 128	80.863285	80.861355	80.863330	0.000024	-1.981291	0.000024	-1.956682	80.863285	80.863285	1.012989e-09	-3.995944	1.012989e-09	1.261409
N = 256	80.863285	80.862799	80.863290	0.000006	-1.990597	0.000006	-1.978148	80.863285	80.863295	1.229648e-07	-4.071341	1.229648e-07	-0.001059
N = 512	80.863285	80.863163	80.863286	0.000002	-1.995285	0.000002	-1.989011	80.863285	80.863291	7.828902e-08	-5.162794	7.828902e-08	-0.000108

Convergence of FDM Temperature at x_bar = 0.637 and k_ratio = 16.0

	True T	FDM 2 T	FDM 2 Extrap T	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 T	FDM 4 Extrap T	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	89.525116	88.797458	NaN	8.127968e-03	NaN	NaN	NaN	89.488625	NaN	4.076008e-04	NaN	NaN	NaN
N = 8	89.525116	89.334419	NaN	2.130095e-03	-1.931977	NaN	NaN	89.522484	NaN	2.939692e-05	-3.793420	NaN	NaN
N = 16	89.525116	89.476830	89.528233	5.741568e-04	-1.981624	5.741568e-04	NaN	89.524945	89.525138	2.154388e-06	-3.944680	2.154388e-06	NaN
N = 32	89.525116	89.513005	89.525323	1.375906e-04	-1.995309	1.375906e-04	-1.977014	89.525105	89.525116	1.244935e-07	-3.985917	1.244935e-07	-3.941793
N = 64	89.525116	89.522085	89.525129	3.399383e-05	-1.998821	3.399383e-05	-1.994135	89.525115	89.525116	7.612812e-09	-3.996463	7.612812e-09	-3.985113
N = 128	89.525116	89.524358	89.525116	8.472662e-06	-1.999705	8.472662e-06	-1.998526	89.525116	89.525116	8.748161e-11	-3.998068	8.748161e-11	-6.320010
N = 256	89.525116	89.524926	89.525116	2.116535e-06	-1.999925	2.116535e-06	-1.999635	89.525116	89.525118	2.897972e-08	-4.004300	2.897972e-08	-0.021888
N = 512	89.525116	89.525068	89.525116	5.290376e-07	-1.999984	5.290376e-07	-1.999904	89.525116	89.525119	3.980727e-08	-1.948540	3.980727e-08	-0.000790

Convergence of FEM Temperature at x_bar = 0.637 and k_ratio = 16.0

	True T	FEM 2 T	FEM 2 Extrap T	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 T	FEM 4 Extrap T	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	89.525116	88.671094	NaN	9.539464e-03	NaN	NaN	NaN	89.520554	NaN	5.095274e-05	NaN	NaN	NaN
N = 8	89.525116	89.272101	NaN	2.826187e-03	-1.755051	NaN	NaN	89.524814	NaN	3.369930e-06	-3.918369	NaN	NaN
N = 16	89.525116	89.455749	89.536559	9.025348e-04	-1.866912	9.025348e-04	NaN	89.525096	89.525117	2.242150e-07	-3.978276	2.242150e-07	NaN
N = 32	89.525116	89.506913	89.526672	2.207033e-04	-1.930093	2.207033e-04	-1.843749	89.525114	89.525116	1.353063e-08	-3.994481	1.353063e-08	-3.982852
N = 64	89.525116	89.520450	89.525321	5.440306e-05	-1.964098	5.440306e-05	-1.918185	89.525116	89.525116	1.848159e-09	-3.998667	1.848159e-09	-2.964279
N = 128	89.525116	89.523934	89.525142	1.348824e-05	-1.981797	1.348824e-05	-1.958049	89.525116	89.525117	1.247827e-08	-4.003756	1.247827e-08	-0.088223
N = 256	89.525116	89.524818	89.525119	3.356780e-06	-1.990833	3.356780e-06	-1.978749	89.525116	89.525130	1.627852e-07	-4.078042	1.627852e-07	-0.000436
N = 512	89.525116	89.525041	89.525116	8.372148e-07	-1.995395	8.372148e-07	-1.989289	89.525116	89.525379	2.942796e-06	-4.742373	2.942796e-06	-0.000001

Section 3 Temp: Convergence of FDM and FEM Temp with Varying k Ratio

Convergence of FDM Heat with x_bar = 0.500 and k_ratio = 0.0625

	True Q	FDM 2 Q	FDM 2 Extrap Q	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 Q	FDM 4 Extrap Q	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	-1.241829	-2.323339	NaN	0.870901	NaN	NaN	NaN	-1.489915	NaN	1.997754e-01	NaN	NaN	NaN
N = 8	-1.241829	-1.583037	NaN	0.274763	-1.664323	NaN	NaN	-1.276591	NaN	2.799300e-02	-2.835241	NaN	NaN
N = 16	-1.241829	-1.335328	-1.210764	0.102880	-1.867617	0.102880	NaN	-1.244804	-1.239238	4.491405e-03	-3.546385	4.491405e-03	NaN
N = 32	-1.241829	-1.265851	-1.238768	0.021863	-1.960577	0.021863	-1.834029	-1.242032	-1.241768	2.131856e-04	-3.867273	2.131856e-04	-3.519703
N = 64	-1.241829	-1.247878	-1.241605	0.005052	-1.989623	0.005052	-1.950668	-1.241842	-1.241827	1.136302e-05	-3.965316	1.136302e-05	-3.860317
N = 128	-1.241829	-1.243344	-1.241814	0.001232	-1.997371	0.001232	-1.987025	-1.241829	-1.241829	6.746119e-07	-3.991275	6.746119e-07	-3.963554
N = 256	-1.241829	-1.242207	-1.241828	0.000306	-1.999340	0.000306	-1.996713	-1.241829	-1.241829	4.157855e-08	-3.998542	4.157855e-08	-3.991048
N = 512	-1.241829	-1.241923	-1.241828	0.000076	-1.999835	0.000076	-1.999175	-1.241829	-1.241829	2.836237e-09	-4.011395	2.836237e-09	-3.875262

Convergence of FEM Heat with x_bar = 0.500 and k_ratio = 0.0625

	True Q	FEM 2 Q	FEM 2 Extrap Q	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 Q	FEM 4 Extrap Q	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	-1.241829	-2.386292	NaN	0.921596	NaN	NaN	NaN	-1.488766	NaN	1.988502e-01	NaN	NaN	NaN
N = 8	-1.241829	-1.638864	NaN	0.319718	-1.527334	NaN	NaN	-1.276827	NaN	2.818331e-02	-2.818769	NaN	NaN
N = 16	-1.241829	-1.364788	-1.206097	0.131575	-1.691080	0.131575	NaN	-1.244914	-1.239256	4.565149e-03	-3.503830	4.565149e-03	NaN
N = 32	-1.241829	-1.276744	-1.235074	0.033739	-1.816262	0.033739	-1.638264	-1.242045	-1.241762	2.281106e-04	-3.829927	2.281106e-04	-3.475886
N = 64	-1.241829	-1.251195	-1.240750	0.008418	-1.898303	0.008418	-1.784977	-1.241843	-1.241827	1.243065e-05	-3.942524	1.243065e-05	-3.821757
N = 128	-1.241829	-1.244259	-1.241674	0.002081	-1.946248	0.002081	-1.881117	-1.241829	-1.241829	7.417125e-07	-3.978996	7.417125e-07	-3.940018
N = 256	-1.241829	-1.242448	-1.241808	0.000515	-1.972331	0.000515	-1.937219	-1.241829	-1.241829	4.576965e-08	-3.992154	4.576965e-08	-3.977715
N = 512	-1.241829	-1.241985	-1.241826	0.000128	-1.985958	0.000128	-1.967700	-1.241829	-1.241829	3.130100e-09	-4.007118	3.130100e-09	-3.864662

Convergence of FDM Heat with x_bar = 0.500 and k_ratio = 0.125

	True Q	FDM 2 Q	FDM 2 Extrap Q	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 Q	FDM 4 Extrap Q	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	-1.756437	-2.635211	NaN	0.500316	NaN	NaN	NaN	-1.898178	NaN	8.069805e-02	NaN	NaN	NaN

```
N = 8   -1.756437 -2.012353          NaN       0.145701      -1.779824                NaN          NaN -1.771468          NaN  8.557588e-03    -3.237258                NaN          NaN
N = 16  -1.756437 -1.823780    -1.741900       0.047006      -1.926064    0.047006          NaN -1.757555    -1.755839  9.773508e-04    -3.748951    9.773508e-04    -3.748951
N = 32  -1.756437 -1.773512    -1.755242       0.010409      -1.979630    0.010409    -1.907407 -1.756511    -1.756426  4.829010e-05    -3.931668    4.829010e-05    -3.735225
N = 64  -1.756437 -1.760722    -1.756356       0.002486      -1.994770    0.002486    -1.974522 -1.756442    -1.756437  2.747175e-06    -3.982525    2.747175e-06    -3.928168
N = 128 -1.756437 -1.756509    -1.756432       0.000613      -1.998684    0.000613    -1.993462 -1.756438    -1.756437  1.671405e-07    -3.995610    1.671405e-07    -3.981650
N = 256 -1.756437 -1.756705    -1.756437       0.000153      -1.999670    0.000153    -1.998354 -1.756437    -1.756437  1.041028e-08    -3.998942    1.041028e-08    -3.990638
N = 512 -1.756437 -1.756504    -1.756437       0.000038      -1.999918    0.000038    -1.999588 -1.756437    -1.756437  1.383606e-09    -4.000125    1.383606e-09    -3.001915

Convergence of FEM Heat with x_bar = 0.500 and k_ratio = 0.125
            True Q   FEM 2 Q  FEM 2 Extrap Q  FEM 2 % Error  FEM 2 Exact B  FEM 2 Extrap % Error  FEM 2 Extrap B   FEM 4 Q  FEM 4 Extrap Q  FEM 4 % Error  FEM 4 Exact B  FEM 4 Extrap % Error  FEM 4 Extrap B
N = 4   -1.756437 -2.724280          NaN       0.551026          NaN                   NaN          NaN -1.897893          NaN  8.053547e-02          NaN                   NaN          NaN
N = 8   -1.756437 -2.072823          NaN       0.180129    -1.613087                   NaN          NaN -1.771784          NaN  8.737232e-03    -3.204376                   NaN          NaN
N = 16  -1.756437 -1.849907    -1.733951       0.066874    -1.759120              0.066874          NaN -1.757613    -1.755819  1.021680e-03    -3.706141    1.021680e-03          NaN
N = 32  -1.756437 -1.782145    -1.752552       0.016886    -1.862264              0.016886    -1.717974 -1.756516    -1.756424  5.242384e-05    -3.901773    5.242384e-05    -3.691040
N = 64  -1.756437 -1.763204    -1.755855       0.004186    -1.925724              0.004186    -1.838899 -1.756442    -1.756437  3.015804e-06    -3.965659    3.015804e-06    -3.897299
N = 128 -1.756437 -1.758175    -1.756357       0.001035    -1.961334              0.001035    -1.913214 -1.756438    -1.756437  1.838668e-07    -3.986774    1.838668e-07    -3.964188
N = 256 -1.756437 -1.756878    -1.756427       0.000257    -1.980259              0.000257    -1.954852 -1.756437    -1.756437  1.141355e-08    -3.994454    1.141355e-08    -3.986272
N = 512 -1.756437 -1.756548    -1.756436       0.000064    -1.990024              0.000064    -1.976959 -1.756437    -1.756437  7.108086e-10    -3.997585    7.108086e-10    -3.996485

Convergence of FDM Heat with x_bar = 0.500 and k_ratio = 0.25
            True Q   FDM 2 Q  FDM 2 Extrap Q  FDM 2 % Error  FDM 2 Exact B  FDM 2 Extrap % Error  FDM 2 Extrap B   FDM 4 Q  FDM 4 Extrap Q  FDM 4 % Error  FDM 4 Exact B  FDM 4 Extrap % Error  FDM 4 Extrap B
N = 4   -2.486946 -3.171745          NaN       0.275357          NaN                   NaN          NaN -2.556595          NaN  2.800566e-02          NaN                   NaN          NaN
N = 8   -2.486946 -2.674530          NaN       0.075427    -1.868146                   NaN          NaN -2.492907          NaN  2.396673e-03    -3.546613                   NaN          NaN
N = 16  -2.486946 -2.538136    -2.480834       0.021889    -1.960724              0.021889          NaN -2.487355    -2.486824  2.132094e-04    -3.867344    2.132094e-04          NaN
N = 32  -2.486946 -2.499080    -2.486500       0.005059    -1.989662              0.005059    -1.950854 -2.486972    -2.486944  1.136517e-05    -3.965333    1.136517e-05    -3.860392
N = 64  -2.486946 -2.489985    -2.486917       0.001234    -1.997380              0.001234    -1.987073 -2.486948    -2.486946  6.747542e-07    -3.991233    6.747542e-07    -3.963582
N = 128 -2.486946 -2.487706    -2.486944       0.000306    -1.999343              0.000306    -1.996725 -2.486946    -2.486946  4.159483e-08    -3.997803    4.159483e-08    -3.990827
N = 256 -2.486946 -2.487136    -2.486946       0.000076    -1.999836              0.000076    -1.999178 -2.486946    -2.486946  2.466575e-09    -3.999493    2.466575e-09    -4.000470
N = 512 -2.486946 -2.486994    -2.486946       0.000019    -1.999959              0.000019    -1.999794 -2.486946    -2.486946  4.903515e-10    -3.999488    4.903515e-10    -1.980183

Convergence of FDM Heat with x_bar = 0.500 and k_ratio = 0.25
            True Q   FEM 2 Q  FEM 2 Extrap Q  FEM 2 % Error  FEM 2 Exact B  FEM 2 Extrap % Error  FEM 2 Extrap B   FEM 4 Q  FEM 4 Extrap Q  FEM 4 % Error  FEM 4 Exact B  FEM 4 Extrap % Error  FEM 4 Extrap B
N = 4   -2.486946 -3.277606          NaN       0.317924          NaN                   NaN          NaN -2.557137          NaN  2.822377e-02          NaN                   NaN          NaN
N = 8   -2.486946 -2.732243          NaN       0.098634    -1.688525                   NaN          NaN -2.493130          NaN  2.486681e-03    -3.504617                   NaN          NaN
N = 16  -2.486946 -2.556840    -2.473243       0.033720    -1.815438              0.033720          NaN -2.487381    -2.486814  2.281691e-04    -3.830158    2.281691e-04          NaN
N = 32  -2.486946 -2.505646    -2.484779       0.008398    -1.897969              0.008398    -1.783949 -2.486974    -2.486944  1.243751e-05    -3.942603    1.243751e-05    -3.822000
N = 64  -2.486946 -2.491799    -2.486637       0.002076    -1.946097              0.002076    -1.880714 -2.486948    -2.486946  7.421638e-07    -3.978987    7.421638e-07    -3.940106
N = 128 -2.486946 -2.488183    -2.486905       0.000514    -1.972258              0.000514    -1.937039 -2.486946    -2.486946  4.576309e-08    -3.991494    4.576309e-08    -3.978786
N = 256 -2.486946 -2.487258    -2.486941       0.000128    -1.985922              0.000128    -1.967615 -2.486946    -2.486946  2.933993e-09    -3.996286    2.933993e-09    -3.952648
N = 512 -2.486946 -2.487025    -2.486946       0.000032    -1.992908              0.000032    -1.983571 -2.486946    -2.486946  1.338225e-09    -3.995277    1.338225e-09    0.015005

Convergence of FDM Heat with x_bar = 0.500 and k_ratio = 0.5
            True Q   FDM 2 Q  FDM 2 Extrap Q  FDM 2 % Error  FDM 2 Exact B  FDM 2 Extrap % Error  FDM 2 Extrap B   FDM 4 Q  FDM 4 Extrap Q  FDM 4 % Error  FDM 4 Exact B  FDM 4 Extrap % Error  FDM 4 Extrap B
N = 4   -3.529489 -4.044925          NaN       0.146037          NaN                   NaN          NaN -3.559595          NaN  8.529659e-03          NaN                   NaN          NaN
N = 8   -3.529489 -3.665089          NaN       0.038419    -1.926443                   NaN          NaN -3.531728          NaN  6.341705e-04    -3.749546                   NaN          NaN
N = 16  -3.529489 -3.563869    -3.527097       0.010426    -1.979723              0.010426          NaN -3.529636    -3.529466  4.809035e-05    -3.931843    4.809035e-05          NaN
N = 32  -3.529489 -3.531115    -3.529327       0.002490    -1.994793              0.002490    -1.974639 -3.529489    -3.529489  2.736381e-06    -3.982671    2.736381e-06    -3.928352
N = 64  -3.529489 -3.531648    -3.529479       0.000614    -1.998689              0.000614    -1.993491 -3.529490    -3.529489  1.664953e-07    -3.995618    1.664953e-07    -3.981696
N = 128 -3.529489 -3.530029    -3.529489       0.000153    -1.999672              0.000153    -1.998362 -3.529489    -3.529489  1.035257e-08    -3.998886    1.035257e-08    -3.992959
N = 256 -3.529489 -3.529624    -3.529489       0.000038    -1.999918              0.000038    -1.999590 -3.529489    -3.529489  1.143815e-09    -3.999052    1.143815e-09    -3.239972
N = 512 -3.529489 -3.529523    -3.529489       0.000010    -1.999979              0.000010    -1.999898 -3.529489    -3.529489  7.826466e-10    -3.962070    7.826466e-10    2.125343

Convergence of FEM Heat with x_bar = 0.500 and k_ratio = 0.5
            True Q   FEM 2 Q  FEM 2 Extrap Q  FEM 2 % Error  FEM 2 Exact B  FEM 2 Extrap % Error  FEM 2 Extrap B   FEM 4 Q  FEM 4 Extrap Q  FEM 4 % Error  FEM 4 Exact B  FEM 4 Extrap % Error  FEM 4 Extrap B
N = 4   -3.529489 -4.151337          NaN       0.176186          NaN                   NaN          NaN -3.560400          NaN  8.757825e-03          NaN                   NaN          NaN
N = 8   -3.529489 -3.714639          NaN       0.052458    -1.747869                   NaN          NaN -3.531856          NaN  6.705386e-04    -3.707180                   NaN          NaN
N = 16  -3.529489 -3.580543    -3.521119       0.016876    -1.858611              0.016876          NaN -3.529648    -3.529462  5.246721e-05    -3.902139    5.246721e-05          NaN
N = 32  -3.529489 -3.529941    -3.528289       0.004153    -1.924207              0.004153    -1.834401 -3.529499    -3.529489  3.019131e-06    -3.965805    3.019131e-06    -3.897682
N = 64  -3.529489 -3.532945    -3.529326       0.001026    -1.960639              0.001026    -1.911393 -3.529490    -3.529489  1.840754e-07    -3.986834    1.840754e-07    -3.964387
N = 128 -3.529489 -3.530365    -3.529468       0.000254    -1.979926              0.000254    -1.954030 -3.529489    -3.529489  1.149619e-08    -3.994426    1.149619e-08    -3.978192
N = 256 -3.529489 -3.529710    -3.529487       0.000063    -1.989861              0.000063    -1.976568 -3.529489    -3.529489  1.754708e-09    -3.996694    1.754708e-09    -2.821280
N = 512 -3.529489 -3.529545    -3.529489       0.000016    -1.994905              0.000016    -1.988168 -3.529489    -3.529489  7.304833e-09    -3.985645    7.304833e-09    0.138062

Convergence of FDM Heat with x_bar = 0.500 and k_ratio = 1.0
            True Q   FDM 2 Q  FDM 2 Extrap Q  FDM 2 % Error  FDM 2 Exact B  FDM 2 Extrap % Error  FDM 2 Extrap B   FDM 4 Q  FDM 4 Extrap Q  FDM 4 % Error  FDM 4 Exact B  FDM 4 Extrap % Error  FDM 4 Extrap B
N = 4   -4.985127 -5.363661          NaN       0.075933          NaN                   NaN          NaN -4.997127          NaN  2.407224e-03          NaN                   NaN          NaN
N = 8   -4.985127 -5.082312          NaN       0.019495    -1.961618                   NaN          NaN -4.985949          NaN  1.649282e-04    -3.867460                   NaN          NaN
N = 16  -4.985127 -5.009594    -4.984248       0.005085    -1.989885              0.005085          NaN -4.985179    -4.985122  1.141325e-05    -3.965359    1.141325e-05          NaN
N = 32  -4.985127 -4.991254    -4.985069       0.001241    -1.997436              0.001241    -1.987353 -4.985130    -4.985126  6.776447e-07    -3.991240    6.776447e-07    -3.963608
N = 64  -4.985127 -4.986659    -4.985123       0.000308    -1.999357              0.000308    -1.996795 -4.985127    -4.985127  4.173827e-08    -3.997804    4.173827e-08    -3.990514
N = 128 -4.985127 -4.985510    -4.985126       0.000077    -1.999839              0.000077    -1.999196 -4.985127    -4.985127  2.828357e-09    -3.999474    2.828357e-09    -3.885074
N = 256 -4.985127 -4.985222    -4.985126       0.000019    -1.999960              0.000019    -1.999799 -4.985127    -4.985127  3.892385e-12    -3.994823    3.892385e-12    -9.291580
N = 512 -4.985127 -4.985150    -4.985127       0.000005    -1.999990              0.000005    -1.999950 -4.985127    -4.985126  1.878544e-08    -4.146631    1.878544e-08    -0.011762

Convergence of FEM Heat with x_bar = 0.500 and k_ratio = 1.0
            True Q   FEM 2 Q  FEM 2 Extrap Q  FEM 2 % Error  FEM 2 Exact B  FEM 2 Extrap % Error  FEM 2 Extrap B   FEM 4 Q  FEM 4 Extrap Q  FEM 4 % Error  FEM 4 Exact B  FEM 4 Extrap % Error  FEM 4 Extrap B
N = 4   -4.985127 -5.459208          NaN       0.095099          NaN                   NaN          NaN -4.997547          NaN  2.491440e-03          NaN                   NaN          NaN
N = 8   -4.985127 -5.122610          NaN       0.027579    -1.785873                   NaN          NaN -4.985999          NaN  1.751167e-04    -3.830591                   NaN          NaN
N = 16  -4.985127 -5.022303    -4.979722       0.008551    -1.886793              0.008551          NaN -4.985183    -4.985121  1.245343e-05    -3.942758    1.245343e-05          NaN
N = 32  -4.985127 -4.994808    -4.984424       0.002083    -1.941161              0.002083    -1.867152 -4.985130    -4.985126  7.431957e-07    -3.979049    7.431957e-07    -3.940265
N = 64  -4.985127 -4.985598    -4.985035       0.000514    -1.969937              0.000514    -1.931164 -4.985127    -4.985127  4.586722e-08    -3.991520    4.586722e-08    -3.977691
N = 128 -4.985127 -4.985751    -4.985115       0.000128    -1.984797              0.000128    -1.964878 -4.985127    -4.985127  3.217920e-09    -3.996280    3.217920e-09    -3.830308
N = 256 -4.985127 -4.985283    -4.985125       0.000032    -1.992354              0.000032    -1.982251 -4.985127    -4.985126  2.619162e-09    -3.995181    2.619162e-09    -1.012964
N = 512 -4.985127 -4.985166    -4.985126       0.000008    -1.996167              0.000008    -1.991076 -4.985127    -4.985127  2.627928e-08    -3.954802    2.627928e-08    0.009191

Convergence of FDM Heat with x_bar = 0.500 and k_ratio = 2.0
            True Q   FDM 2 Q  FDM 2 Extrap Q  FDM 2 % Error  FDM 2 Exact B  FDM 2 Extrap % Error  FDM 2 Extrap B   FDM 4 Q  FDM 4 Extrap Q  FDM 4 % Error  FDM 4 Exact B  FDM 4 Extrap % Error  FDM 4 Extrap B
N = 4   -6.832006 -7.120389          NaN       0.042211          NaN                   NaN          NaN -6.837657          NaN  8.271733e-04          NaN                   NaN          NaN
N = 8   -6.832006 -6.905215          NaN       0.010716    -1.977901                   NaN          NaN -6.832382          NaN  5.498761e-05    -3.911011                   NaN          NaN
N = 16  -6.832006 -6.850381    -6.831629       0.002745    -1.994255              0.002745          NaN -6.832030    -6.832005  3.679505e-06    -3.976922    3.679505e-06          NaN
N = 32  -6.832006 -6.836605    -6.831982       0.000677    -1.998549              0.000677    -1.992819 -6.832008    -6.832006  2.221497e-07    -3.994176    2.221497e-07    -3.975751
N = 64  -6.832006 -6.833156    -6.832005       0.000169    -1.999636              0.000169    -1.998186 -6.832006    -6.832006  1.379322e-08    -3.998527    1.379322e-08    -3.990379
N = 128 -6.832006 -6.832294    -6.832006       0.000042    -1.999909              0.000042    -1.999549 -6.832006    -6.832006  1.862709e-09    -3.999120    1.862709e-09    -2.981077
N = 256 -6.832006 -6.832078    -6.832006       0.000011    -1.999977              0.000011    -1.999886 -6.832006    -6.832006  1.324195e-08    -3.970921    1.324195e-08    0.090221
N = 512 -6.832006 -6.832024    -6.832006       0.000003    -1.999997              0.000003    -1.999971 -6.832006    -6.832005  2.031033e-07    -2.712521    2.031033e-07    -0.000329

Convergence of FEM Heat with x_bar = 0.500 and k_ratio = 2.0
            True Q   FEM 2 Q  FEM 2 Extrap Q  FEM 2 % Error  FEM 2 Exact B  FEM 2 Extrap % Error  FEM 2 Extrap B   FEM 4 Q  FEM 4 Extrap Q  FEM 4 % Error  FEM 4 Exact B  FEM 4 Extrap % Error  FEM 4 Extrap B
N = 4   -6.832006 -7.202967          NaN       0.054297          NaN                   NaN          NaN -6.836734          NaN  6.919495e-04          NaN                   NaN          NaN
N = 8   -6.832006 -6.935575          NaN       0.015613    -1.798121                   NaN          NaN -6.832322          NaN  4.628007e-05    -3.902204                   NaN          NaN
N = 16  -6.832006 -6.860626    -6.827918       0.004790    -1.898047              0.004790          NaN -6.832026    -6.832005  3.115230e-06    -3.965787    3.115230e-06          NaN
N = 32  -6.832006 -6.839422    -6.831512       0.001158    -1.948400              0.001158    -1.880015 -6.832007    -6.832006  1.899456e-07    -3.986822    1.899456e-07    -3.964396
N = 64  -6.832006 -6.833894    -6.831944       0.000285    -1.974008              0.000285    -1.939550 -6.832006    -6.832006  1.170627e-08    -3.994421    1.170627e-08    -3.996168
N = 128 -6.832006 -6.832482    -6.831998       0.000071    -1.986952              0.000071    -1.969611 -6.832006    -6.832006  4.876349e-10    -3.998529    4.876349e-10    -4.288628
N = 256 -6.832006 -6.832126    -6.832005       0.000018    -1.993462              0.000018    -1.984762 -6.832006    -6.832006  4.071656e-09    -3.990079    4.071656e-09    0.267031
N = 512 -6.832006 -6.832036    -6.832006       0.000004    -1.996727              0.000004    -1.992369 -6.832006    -6.832006  5.252254e-08    -4.462020    5.252254e-08    0.001327

Convergence of FDM Heat with x_bar = 0.500 and k_ratio = 4.0
            True Q   FDM 2 Q  FDM 2 Extrap Q  FDM 2 % Error  FDM 2 Exact B  FDM 2 Extrap % Error  FDM 2 Extrap B   FDM 4 Q  FDM 4 Extrap Q  FDM 4 % Error  FDM 4 Exact B  FDM 4 Extrap % Error  FDM 4 Extrap B
N = 4   -8.797049 -9.060288          NaN       0.029924          NaN                   NaN          NaN -8.802357          NaN  6.033795e-04          NaN                   NaN          NaN
N = 8   -8.797049 -8.872016          NaN       0.007612    -1.974855                   NaN          NaN -8.797607          NaN  4.074871e-05    -3.888310                   NaN          NaN
N = 16  -8.797049 -8.813868    -8.796658       0.001956    -1.993338              0.001956          NaN -8.797072    -8.797047  2.774659e-06    -3.970790    2.774659e-06          NaN
N = 32  -8.797049 -8.801259    -8.797023       0.000481    -1.998309              0.000481    -1.991674 -8.797050    -8.797049  1.660966e-07    -3.992612    1.660966e-07    -3.969303
N = 64  -8.797049 -8.798102    -8.797047       0.000120    -1.999576              0.000120    -1.997886 -8.797049    -8.797049  1.028146e-08    -3.998156    1.028146e-08    -3.989594
N = 128 -8.797049 -8.797312    -8.797049       0.000030    -1.999894              0.000030    -1.999469 -8.797049    -8.797049  4.932746e-10    -3.998186    4.932746e-10    -4.202856
N = 256 -8.797049 -8.797115    -8.797049       0.000007    -1.999973              0.000007    -1.999867 -8.797049    -8.797049  1.765533e-09    -4.027343    1.765533e-09    -0.422141
N = 512 -8.797049 -8.797065    -8.797049       0.000002    -1.999997              0.000002    -1.999954 -8.797049    -8.797046  2.985972e-07    -3.904864    2.985972e-07    -0.000177

Convergence of FEM Heat with x_bar = 0.500 and k_ratio = 4.0
            True Q   FEM 2 Q  FEM 2 Extrap Q  FEM 2 % Error  FEM 2 Exact B  FEM 2 Extrap % Error  FEM 2 Extrap B   FEM 4 Q  FEM 4 Extrap Q  FEM 4 % Error  FEM 4 Exact B  FEM 4 Extrap % Error  FEM 4 Extrap B
N = 4   -8.797049 -9.133348          NaN       0.038229          NaN                   NaN          NaN -8.799070          NaN  2.297472e-04          NaN                   NaN          NaN
N = 8   -8.797049 -8.894031          NaN       0.011024    -1.793956                   NaN          NaN -8.797180          NaN  1.494372e-05    -3.942436                   NaN          NaN
N = 16  -8.797049 -8.803792    -8.793202       0.003399    -1.896890              0.003399          NaN -8.797057    -8.797049  9.755775e-07    -3.979737    9.755775e-07          NaN
N = 32  -8.797049 -8.803797    -8.796590       0.000819    -1.948194              0.000819    -1.878506 -8.797049    -8.797049  6.012137e-08    -3.992123    6.012137e-08    -3.978619
N = 64  -8.797049 -8.798767    -8.796992       0.000202    -1.974011              0.000202    -1.939272 -8.797049    -8.797049  3.963173e-09    -3.996647    3.963173e-09    -3.914403
N = 128 -8.797049 -8.797482    -8.797042       0.000050    -1.986981              0.000050    -1.969609 -8.797049    -8.797049  2.397599e-09    -3.996510    2.397599e-09    1.127149
N = 256 -8.797049 -8.797158    -8.797048       0.000012    -1.993485              0.000012    -1.984794 -8.797049    -8.797049  6.950475e-08    -3.965261    6.950475e-08    -0.004532
N = 512 -8.797049 -8.797076    -8.797049       0.000003    -1.996745              0.000003    -1.992394 -8.797049    -8.797060  1.303768e-06    -1.647698    1.303768e-06    0.000022

Convergence of FDM Heat with x_bar = 0.500 and k_ratio = 8.0
            True Q   FDM 2 Q  FDM 2 Extrap Q  FDM 2 % Error  FDM 2 Exact B  FDM 2 Extrap % Error  FDM 2 Extrap B   FDM 4 Q  FDM 4 Extrap Q  FDM 4 % Error  FDM 4 Exact B  FDM 4 Extrap % Error  FDM 4 Extrap B
N = 4   -10.486214 -10.779857          NaN       0.028003          NaN                   NaN          NaN -10.493612          NaN  7.054714e-04          NaN                   NaN          NaN
N = 8   -10.486214 -10.561319          NaN       0.007162    -1.967076                   NaN          NaN -10.486719          NaN  4.814601e-05    -3.873099                   NaN          NaN
N = 16  -10.486214 -10.486104    -10.485636       0.001857    -1.991276              0.001857          NaN -10.486246    -10.486211  3.317272e-06    -3.966863    3.317279e-06          NaN
N = 32  -10.486214 -10.490944    -10.486176       0.000455    -1.997786              0.000455    -1.989096 -10.486216    -10.486214  1.973876e-07    -3.991621    1.973876e-07    -3.965170
N = 64  -10.486214 -10.487397    -10.486211       0.000113    -1.999444              0.000113    -1.997232 -10.486214    -10.486214  1.213030e-08    -3.997866    1.213030e-08    -3.996212
N = 128 -10.486214 -10.486510    -10.486214       0.000028    -1.999861              0.000028    -1.999305 -10.486214    -10.486214  2.131372e-09    -3.997144    2.131372e-09    -2.661836
N = 256 -10.486214 -10.486288    -10.486214       0.000007    -1.999966              0.000007    -1.999826 -10.486214    -10.486214  4.274711e-09    -3.846239    4.274711e-09    -0.220515
N = 512 -10.486214 -10.486232    -10.486214       0.000002    -1.999975              0.000002    -1.999969 -10.486214    -10.486214  4.898006e-09    -0.513431    4.898006e-09    -0.004645

Convergence of FEM Heat with x_bar = 0.500 and k_ratio = 8.0
            True Q   FEM 2 Q  FEM 2 Extrap Q  FEM 2 % Error  FEM 2 Exact B  FEM 2 Extrap % Error  FEM 2 Extrap B   FEM 4 Q  FEM 4 Extrap Q  FEM 4 % Error  FEM 4 Exact B  FEM 4 Extrap % Error  FEM 4 Extrap B
N = 4   -10.486214 -10.847776          NaN       0.034480          NaN                   NaN          NaN -10.487582          NaN  1.304358e-04          NaN                   NaN          NaN
N = 8   -10.486214 -10.590389          NaN       0.009935    -1.795226                   NaN          NaN -10.486302          NaN  8.398688e-06    -3.957032                   NaN          NaN
N = 16  -10.486214 -10.514219    -10.482202       0.003054    -1.895250              0.003054          NaN -10.486219    -10.486214  5.423604e-07    -3.986778    5.423604e-07          NaN
N = 32  -10.486214 -10.493478    -10.485717       0.000740    -1.946777              0.000740    -1.876758 -10.486214    -10.486214  3.340698e-08    -3.995622    3.340698e-08    -3.987629
N = 64  -10.486214 -10.488064    -10.486151       0.000182    -1.973142              0.000182    -1.937655 -10.486214    -10.486214  2.688212e-09    -3.998136    2.688212e-09    -3.653156
N = 128 -10.486214 -10.486681    -10.486206       0.000045    -1.986505              0.000045    -1.968604 -10.486214    -10.486214  6.990587e-09    -3.995967    6.990587e-09    -0.354642
N = 256 -10.486214 -10.486331    -10.486213       0.000011    -1.993235              0.000011    -1.984241 -10.486214    -10.486213  8.708136e-08    -5.849693    8.708136e-08    -0.002119
N = 512 -10.486214 -10.486243    -10.486214       0.000003    -1.996597              0.000003    -1.992109 -10.486214    -10.486224  9.472036e-07    3.111172     9.472036e-07    -0.000026

Convergence of FDM Heat with x_bar = 0.500 and k_ratio = 16.0
            True Q   FDM 2 Q  FDM 2 Extrap Q  FDM 2 % Error  FDM 2 Exact B  FDM 2 Extrap % Error  FDM 2 Extrap B   FDM 4 Q  FDM 4 Extrap Q  FDM 4 % Error  FDM 4 Exact B  FDM 4 Extrap % Error  FDM 4 Extrap B
N = 4   -11.680625 -12.024326          NaN       0.029425          NaN                   NaN          NaN -11.690359          NaN  8.333062e-04          NaN                   NaN          NaN
N = 8   -11.680625 -11.768665          NaN       0.007537    -1.964919                   NaN          NaN -11.681291          NaN  5.699702e-05    -3.869888                   NaN          NaN
N = 16  -11.680625 -11.702777    -11.679901       0.001959    -1.990732              0.001959          NaN -11.680668    -11.680622  3.936877e-06    -3.966020    3.936877e-06          NaN
N = 32  -11.680625 -11.686172    -11.680577       0.000479    -1.997649              0.000479    -1.988414 -11.680628    -11.680625  2.339626e-07    -3.991409    2.339626e-07    -3.964284
N = 64  -11.680625 -11.682012    -11.680622       0.000119    -1.999410              0.000119    -1.997062 -11.680625    -11.680625  1.444745e-08    -3.997772    1.444745e-08    -3.989014
N = 128 -11.680625 -11.680972    -11.680625       0.000030    -1.999852              0.000030    -1.999263 -11.680625    -11.680625  1.394570e-09    -4.001136    1.394570e-09    -3.412796
N = 256 -11.680625 -11.680712    -11.680625       0.000007    -1.999964              0.000007    -1.999815 -11.680625    -11.680625  1.645965e-08    -3.981056    1.645965e-08    -0.071781
N = 512 -11.680625 -11.680647    -11.680625       0.000002    -1.999977              0.000002    -1.999958 -11.680625    -11.680624  1.365727e-07    0.019875     1.365727e-07    0.000008

Convergence of FEM Heat with x_bar = 0.500 and k_ratio = 16.0
```

	True Q	FEM 2 Q	FEM 2 Extrap Q	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 Q	FEM 4 Extrap Q	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	-11.680625	-12.090051	NaN	0.035052	NaN	NaN	NaN	-11.682009	NaN	1.185027e-04	NaN	NaN	NaN
N = 8	-11.680625	-11.797909	NaN	0.010041	-1.803594	NaN	NaN	-11.680714	NaN	7.625382e-06	-3.957966	NaN	NaN
N = 16	-11.680625	-11.712125	-11.676464	0.003054	-1.896582	0.003054	NaN	-11.680631	-11.680625	4.921369e-07	-3.988478	4.921369e-07	NaN
N = 32	-11.680625	-11.688797	-11.680083	0.000746	-1.946658	0.000746	-1.878621	-11.680625	-11.680625	3.029433e-08	-3.996830	3.029433e-08	-3.987749
N = 64	-11.680625	-11.682707	-11.680555	0.000184	-1.972873	0.000184	-1.937587	-11.680625	-11.680625	1.896672e-09	-3.998848	1.896672e-09	-4.405824
N = 128	-11.680625	-11.681150	-11.680616	0.000046	-1.986316	0.000046	-1.968306	-11.680625	-11.680625	4.998899e-09	-3.994560	4.998899e-09	-0.436061
N = 256	-11.680625	-11.680757	-11.680624	0.000011	-1.993128	0.000011	-1.984024	-11.680625	-11.680624	9.229468e-08	-6.569744	9.229468e-08	-0.001825
N = 512	-11.680625	-11.680658	-11.680625	0.000003	-1.996572	0.000003	-1.991978	-11.680625	-11.680638	1.106963e-06	4.036521	1.106963e-06	-0.000025

Convergence of FDM Heat with x_bar = 0.637 and k_ratio = 0.0625

	True Q	FDM 2 Q	FDM 2 Extrap Q	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 Q	FDM 4 Extrap Q	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	-1.241978	-1.892412	NaN	0.523708	NaN	NaN	NaN	-1.350272	NaN	8.719507e-02	NaN	NaN	NaN
N = 8	-1.241978	-1.432463	NaN	0.153372	-1.771728	NaN	NaN	-1.253695	NaN	9.433953e-03	-3.208312	NaN	NaN
N = 16	-1.241978	-1.292229	-1.230720	0.049978	-1.922447	0.049978	NaN	-1.242857	-1.241487	1.103433e-03	-3.736406	1.103433e-03	NaN
N = 32	-1.241978	-1.254729	-1.241041	0.011030	-1.978528	0.011030	-1.902870	-1.242036	-1.241968	5.422868e-05	-3.927976	5.422868e-05	-3.721927
N = 64	-1.241978	-1.245178	-1.241914	0.002628	-1.994480	0.002628	-1.973144	-1.241982	-1.241978	3.071666e-06	-3.982155	3.071666e-06	-3.924241
N = 128	-1.241978	-1.242779	-1.241974	0.000648	-1.998612	0.000648	-1.993098	-1.241978	-1.241978	1.866167e-07	-4.005072	1.866167e-07	-3.980600
N = 256	-1.241978	-1.242178	-1.241978	0.000161	-1.999661	0.000161	-1.998262	-1.241978	-1.241978	1.167171e-08	-4.163740	1.167171e-08	-3.984220
N = 512	-1.241978	-1.242028	-1.241978	0.000040	-1.999948	0.000040	-1.999565	-1.241978	-1.241978	1.129908e-09	-4.093738	1.129908e-09	-3.403237

Convergence of FEM Heat with x_bar = 0.637 and k_ratio = 0.0625

	True Q	FEM 2 Q	FEM 2 Extrap Q	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 Q	FEM 4 Extrap Q	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	-1.241978	-1.955601	NaN	0.574586	NaN	NaN	NaN	-1.350026	NaN	8.698070e-02	NaN	NaN	NaN
N = 8	-1.241978	-1.472318	NaN	0.185683	-1.606560	NaN	NaN	-1.253926	NaN	9.620105e-03	-3.176571	NaN	NaN
N = 16	-1.241978	-1.311448	-1.224995	0.070574	-1.754142	0.070574	NaN	-1.242902	-1.241473	1.150974e-03	-3.693351	1.150974e-03	NaN
N = 32	-1.241978	-1.261128	-1.239024	0.017840	-1.859012	0.017840	-1.712139	-1.242040	-1.241967	5.881873e-05	-3.897482	5.881873e-05	-3.677491
N = 64	-1.241978	-1.247025	-1.241533	0.004424	-1.923832	0.004424	-1.835089	-1.241982	-1.241978	3.371315e-06	-3.964820	3.371315e-06	-3.892758
N = 128	-1.241978	-1.243275	-1.241917	0.001094	-1.960308	0.001094	-1.910998	-1.241978	-1.241978	2.052741e-07	-3.995169	2.052741e-07	-3.962755
N = 256	-1.241978	-1.242307	-1.241970	0.000271	-1.979729	0.000271	-1.953652	-1.241978	-1.241978	1.268251e-08	-4.143718	1.268251e-08	-3.991678
N = 512	-1.241978	-1.242061	-1.241977	0.000067	-1.989772	0.000067	-1.976334	-1.241978	-1.241978	1.753379e-09	-4.424299	1.753379e-09	-2.957447

Convergence of FEM Heat with x_bar = 0.637 and k_ratio = 0.125

	True Q	FDM 2 Q	FDM 2 Extrap Q	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 Q	FDM 4 Extrap Q	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	-1.758799	-2.268096	NaN	0.289571	NaN	NaN	NaN	-1.812674	NaN	3.063204e-02	NaN	NaN	NaN
N = 8	-1.758799	-1.898803	NaN	0.079692	-1.863040	NaN	NaN	-1.763475	NaN	2.688650e-03	-3.526276	NaN	NaN
N = 16	-1.758799	-1.794813	-1.754052	0.023238	-1.958837	0.023238	NaN	-1.759121	-1.758698	2.403705e-04	-3.860338	2.403705e-04	NaN
N = 32	-1.758799	-1.767870	-1.758449	0.005358	-1.989132	0.005358	-1.948491	-1.758819	-1.758797	1.273760e-05	-3.963414	1.273760e-05	-3.853001
N = 64	-1.758799	-1.761071	-1.758776	0.001305	-1.997244	0.001305	-1.986411	-1.758800	-1.758799	7.542667e-07	-3.990746	7.542667e-07	-3.961560
N = 128	-1.758799	-1.759367	-1.758797	0.000324	-1.999308	0.000324	-1.996554	-1.758799	-1.758799	4.646868e-08	-3.997740	4.646868e-08	-3.990090
N = 256	-1.758799	-1.758941	-1.758799	0.000081	-1.999827	0.000081	-1.999136	-1.758799	-1.758799	2.749222e-09	-4.000295	2.749222e-09	-4.066683
N = 512	-1.758799	-1.758834	-1.758799	0.000020	-1.999957	0.000020	-1.999784	-1.758799	-1.758799	1.148633e-09	-4.020561	1.148633e-09	-0.442350

Convergence of FEM Heat with x_bar = 0.637 and k_ratio = 0.125

	True Q	FEM 2 Q	FEM 2 Extrap Q	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 Q	FEM 4 Extrap Q	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	-1.758799	-2.343653	NaN	0.332530	NaN	NaN	NaN	-1.813018	NaN	3.082735e-02	NaN	NaN	NaN
N = 8	-1.758799	-1.940923	NaN	0.103550	-1.683153	NaN	NaN	-1.763642	NaN	2.753720e-03	-3.484757	NaN	NaN
N = 16	-1.758799	-1.811685	-1.748438	0.035602	-1.811494	0.035602	NaN	-1.759141	-1.758590	2.568924e-04	-3.822534	2.568924e-04	NaN
N = 32	-1.758799	-1.772745	-1.757148	0.008876	-1.895524	0.008876	-1.779334	-1.758821	-1.758797	1.392563e-05	-3.940173	1.392563e-05	-3.814075
N = 64	-1.758799	-1.762421	-1.758563	0.002194	-1.944724	0.002194	-1.877853	-1.758800	-1.758799	8.290856e-07	-3.978186	8.290856e-07	-3.937561
N = 128	-1.758799	-1.759723	-1.758767	0.000543	-1.971529	0.000543	-1.935435	-1.758799	-1.758799	5.110725e-08	-3.991256	5.110725e-08	-3.977434
N = 256	-1.758799	-1.759032	-1.758795	0.000135	-1.985546	0.000135	-1.966760	-1.758799	-1.758799	3.091361e-09	-3.997014	3.091361e-09	-4.030266
N = 512	-1.758799	-1.758857	-1.758798	0.000034	-1.992717	0.000034	-1.983132	-1.758799	-1.758799	3.903887e-11	-4.015170	3.903887e-11	-6.231455

Convergence of FEM Heat with x_bar = 0.637 and k_ratio = 0.25

	True Q	FDM 2 Q	FDM 2 Extrap Q	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 Q	FDM 4 Extrap Q	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	-2.501202	-2.888923	NaN	0.155014	NaN	NaN	NaN	-2.524883	NaN	9.467815e-03	NaN	NaN	NaN
N = 8	-2.501202	-2.603336	NaN	0.040834	-1.924555	NaN	NaN	-2.502977	NaN	7.099598e-04	-3.737222	NaN	NaN
N = 16	-2.501202	-2.527108	-2.499354	0.011105	-1.979071	0.011105	NaN	-2.501318	-2.501182	5.435757e-05	-3.928158	5.435757e-05	NaN
N = 32	-2.501202	-2.507703	-2.501075	0.002650	-1.994616	0.002650	-1.973826	-2.501209	-2.501201	3.079948e-06	-3.981605	3.079948e-06	-3.924474
N = 64	-2.501202	-2.502828	-2.501194	0.000654	-1.998644	0.000654	-1.993269	-2.501202	-2.501202	1.871398e-07	-3.995374	1.871398e-07	-3.980634
N = 128	-2.501202	-2.501608	-2.501201	0.000163	-1.999660	0.000163	-1.998305	-2.501202	-2.501202	1.160713e-08	-3.998840	1.160713e-08	-3.995524
N = 256	-2.501202	-2.501303	-2.501202	0.000041	-1.999915	0.000041	-1.999576	-2.501202	-2.501202	3.385139e-10	-3.999523	3.385139e-10	-4.957265
N = 512	-2.501202	-2.501227	-2.501202	0.000010	-1.999979	0.000010	-1.999894	-2.501202	-2.501202	2.176780e-08	-4.006309	2.176780e-08	-0.044305

Convergence of FEM Heat with x_bar = 0.637 and k_ratio = 0.25

	True Q	FDM 2 Q	FDM 2 Extrap Q	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 Q	FDM 4 Extrap Q	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	-2.501202	-2.964641	NaN	0.185287	NaN	NaN	NaN	-2.525373	NaN	9.664044e-03	NaN	NaN	NaN
N = 8	-2.501202	-2.639437	NaN	0.055267	-1.745259	NaN	NaN	-2.503068	NaN	7.461252e-04	-3.695137	NaN	NaN
N = 16	-2.501202	-2.539384	-2.494922	0.017821	-1.856151	0.017821	NaN	-2.501327	-2.501179	5.893199e-05	-3.898067	5.893199e-05	NaN
N = 32	-2.501202	-2.512273	-2.500289	0.004393	-1.922655	0.004393	-1.831552	-2.501210	-2.501201	3.379471e-06	-3.964507	3.379471e-06	-3.896606
N = 64	-2.501202	-2.503791	-2.501077	0.001085	-1.959773	0.001085	-1.909585	-2.501202	-2.501202	2.057958e-07	-3.986394	2.057958e-07	-3.963003
N = 128	-2.501202	-2.501858	-2.501185	0.000269	-1.979469	0.000269	-1.953020	-2.501202	-2.501202	1.273715e-08	-3.994267	1.273715e-08	-3.989443
N = 256	-2.501202	-2.501367	-2.501200	0.000067	-1.989627	0.000067	-1.976035	-2.501202	-2.501202	1.639429e-09	-3.997049	1.639429e-09	-3.045573
N = 512	-2.501202	-2.501243	-2.501201	0.000017	-1.994786	0.000017	-1.987895	-2.501202	-2.501202	9.024079e-09	-4.008757	9.024079e-09	-0.114514

Convergence of FEM Heat with x_bar = 0.637 and k_ratio = 0.5

	True Q	FDM 2 Q	FDM 2 Extrap Q	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 Q	FDM 4 Extrap Q	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	-3.564213	-3.858649	NaN	0.082609	NaN	NaN	NaN	-3.574224	NaN	2.808695e-03	NaN	NaN	NaN
N = 8	-3.564213	-3.639854	NaN	0.021222	-1.960716	NaN	NaN	-3.564905	NaN	1.940781e-04	-3.855191	NaN	NaN
N = 16	-3.564213	-3.583260	-3.563514	0.005541	-1.989586	0.005541	NaN	-3.564257	-3.564209	1.355828e-05	-3.961911	1.355828e-05	NaN
N = 32	-3.564213	-3.568383	-3.564167	0.001351	-1.997356	0.001351	-1.986981	-3.564216	-3.564213	8.014377e-07	-3.990351	8.014377e-07	-3.959892
N = 64	-3.564213	-3.565406	-3.564210	0.000336	-1.999336	0.000336	-1.996695	-3.564213	-3.564213	4.932485e-08	-3.997581	4.932485e-08	-3.990263
N = 128	-3.564213	-3.564511	-3.564213	0.000084	-1.999834	0.000084	-1.999170	-3.564213	-3.564213	2.989209e-09	-3.999323	2.989209e-09	-4.034364
N = 256	-3.564213	-3.564287	-3.564213	0.000021	-1.999958	0.000021	-1.999792	-3.564213	-3.564213	4.437690e-09	-4.004355	4.437690e-09	-0.720881
N = 512	-3.564213	-3.564231	-3.564213	0.000005	-1.999990	0.000005	-1.999949	-3.564213	-3.564213	1.230940e-08	-3.892410	1.230940e-08	0.021047

Convergence of FEM Heat with x_bar = 0.637 and k_ratio = 0.5

	True Q	FDM 2 Q	FDM 2 Extrap Q	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 Q	FDM 4 Extrap Q	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	-3.564213	-3.926446	NaN	0.101631	NaN	NaN	NaN	-3.574143	NaN	2.786130e-03	NaN	NaN	NaN
N = 8	-3.564213	-3.669207	NaN	0.029458	-1.786611	NaN	NaN	-3.564914	NaN	1.966313e-04	-3.824698	NaN	NaN
N = 16	-3.564213	-3.592628	-3.560167	0.009118	-1.885584	0.009118	NaN	-3.564258	-3.564208	1.404724e-05	-3.940985	1.404724e-05	NaN
N = 32	-3.564213	-3.571617	-3.563673	0.002229	-1.940239	0.002229	-1.865819	-3.564216	-3.564213	8.366621e-07	-3.978534	8.366621e-07	-3.938406
N = 64	-3.564213	-3.566142	-3.564152	0.000550	-1.969405	0.000550	-1.930100	-3.564213	-3.564213	5.159298e-08	-3.991362	5.159298e-08	-3.977450
N = 128	-3.564213	-3.564691	-3.564204	0.000137	-1.984514	0.000137	-1.964260	-3.564213	-3.564213	3.111179e-09	-3.996222	3.111179e-09	-4.034282
N = 256	-3.564213	-3.564333	-3.564212	0.000034	-1.992209	0.000034	-1.981922	-3.564213	-3.564213	5.160476e-09	-3.999598	5.160476e-09	-0.661124
N = 512	-3.564213	-3.564243	-3.564213	0.000008	-1.996091	0.000008	-1.990908	-3.564213	-3.564213	4.864156e-08	-3.819867	4.864156e-08	0.005524

Convergence of FEM Heat with x_bar = 0.637 and k_ratio = 1.0

	True Q	FDM 2 Q	FDM 2 Extrap Q	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 Q	FDM 4 Extrap Q	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	-4.985127	-5.229953	NaN	0.049111	NaN	NaN	NaN	-4.991324	NaN	1.243210e-03	NaN	NaN	NaN
N = 8	-4.985127	-5.047653	NaN	0.012543	-1.969225	NaN	NaN	-4.985553	NaN	8.562086e-05	-3.859964	NaN	NaN
N = 16	-4.985127	-5.000849	-4.984681	0.003243	-1.991676	0.003243	NaN	-4.985154	-4.985124	5.959042e-06	-3.962550	5.959042e-06	NaN
N = 32	-4.985127	-4.989063	-4.985096	0.000796	-1.997874	0.000796	-1.989601	-4.985128	-4.985126	3.532359e-07	-3.990470	3.532359e-07	-3.960660
N = 64	-4.985127	-4.986111	-4.985125	0.000198	-1.999466	0.000198	-1.997343	-4.985127	-4.985127	2.177549e-08	-3.997603	2.177549e-08	-3.988623
N = 128	-4.985127	-4.985373	-4.985126	0.000049	-1.999866	0.000049	-1.999332	-4.985127	-4.985127	1.004778e-09	-3.999222	1.004778e-09	-4.404782
N = 256	-4.985127	-4.985188	-4.985127	0.000012	-1.999967	0.000012	-1.999833	-4.985127	-4.985127	1.382190e-08	-4.001567	1.382190e-08	0.138848
N = 512	-4.985127	-4.985142	-4.985127	0.000003	-1.999989	0.000003	-1.999960	-4.985127	-4.985127	9.249491e-09	-4.239600	9.249491e-09	0.011826

Convergence of FEM Heat with x_bar = 0.637 and k_ratio = 1.0

	True Q	FDM 2 Q	FDM 2 Extrap Q	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 Q	FDM 4 Extrap Q	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	-4.985127	-5.290994	NaN	0.061356	NaN	NaN	NaN	-4.989179	NaN	8.128531e-04	NaN	NaN	NaN
N = 8	-4.985127	-5.073389	NaN	0.017705	-1.793026	NaN	NaN	-4.985398	NaN	5.448497e-05	-3.899064	NaN	NaN
N = 16	-4.985127	-5.008894	-4.981727	0.005453	-1.892797	0.005453	NaN	-4.985144	-4.985126	3.676566e-06	-3.965409	3.676566e-06	NaN
N = 32	-4.985127	-4.991298	-4.984696	0.001324	-1.945278	0.001324	-1.873929	-4.985128	-4.985126	2.237553e-07	-3.986976	2.237553e-07	-3.963947
N = 64	-4.985127	-4.986699	-4.985072	0.000326	-1.972350	0.000326	-1.935902	-4.985127	-4.985127	1.393229e-08	-3.994584	1.393229e-08	-3.981692
N = 128	-4.985127	-4.985524	-4.985120	0.000081	-1.986102	0.000081	-1.967677	-4.985127	-4.985127	8.685149e-11	-3.997184	8.685149e-11	-7.228264
N = 256	-4.985127	-4.985226	-4.985126	0.000020	-1.993033	0.000020	-1.983770	-4.985127	-4.985126	1.520828e-08	-4.005209	1.520828e-08	-0.074896
N = 512	-4.985127	-4.985152	-4.985126	0.000005	-1.996511	0.000005	-1.991868	-4.985127	-4.985126	1.693590e-07	-5.484669	1.693590e-07	-0.000448

Convergence of FEM Heat with x_bar = 0.637 and k_ratio = 2.0

	True Q	FDM 2 Q	FDM 2 Extrap Q	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 Q	FDM 4 Extrap Q	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	-6.610807	-6.872675	NaN	0.039613	NaN	NaN	NaN	-6.619428	NaN	1.304078e-03	NaN	NaN	NaN
N = 8	-6.610807	-6.687556	NaN	0.010248	-1.950595	NaN	NaN	-6.611422	NaN	8.300072e-05	-3.809641	NaN	NaN
N = 16	-6.610807	-6.627905	-6.610024	0.002705	-1.986349	0.002705	NaN	-6.610847	-6.610802	6.730247e-06	-3.949086	6.730247e-06	NaN
N = 32	-6.610807	-6.615092	-6.610753	0.000656	-1.996494	0.000656	-1.982940	-6.610809	-6.610807	3.913050e-07	-3.987045	3.913050e-07	-3.946493
N = 64	-6.610807	-6.611879	-6.610804	0.000163	-1.999117	0.000163	-1.995618	-6.610807	-6.610807	2.399182e-08	-3.996738	2.399182e-08	-3.985134
N = 128	-6.610807	-6.611075	-6.610807	0.000041	-1.999779	0.000041	-1.998897	-6.610807	-6.610807	1.565348e-09	-3.999742	1.565348e-09	-3.969495
N = 256	-6.610807	-6.610874	-6.610807	0.000010	-1.999945	0.000010	-1.999724	-6.610807	-6.610807	8.518639e-09	-3.981280	8.518639e-09	-0.218471
N = 512	-6.610807	-6.610824	-6.610807	0.000003	-1.999987	0.000003	-1.999931	-6.610807	-6.610807	7.573610e-08	-4.739885	7.573610e-08	0.001727

Convergence of FEM Heat with x_bar = 0.637 and k_ratio = 2.0

	True Q	FDM 2 Q	FDM 2 Extrap Q	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 Q	FDM 4 Extrap Q	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	-6.610807	-6.934589	NaN	0.048978	NaN	NaN	NaN	-6.613014	NaN	3.338215e-04	NaN	NaN	NaN
N = 8	-6.610807	-6.698769	NaN	0.014360	-1.769597	NaN	NaN	-6.610952	NaN	2.191865e-05	-3.928846	NaN	NaN
N = 16	-6.610807	-6.636589	-6.606609	0.004538	-1.881011	0.004538	NaN	-6.610816	-6.610807	1.446114e-06	-3.977416	1.446114e-06	NaN
N = 32	-6.610807	-6.617529	-6.610280	0.001097	-1.939466	0.001097	-1.859818	-6.610808	-6.610807	8.847010e-08	-3.992259	8.847010e-08	-3.976406
N = 64	-6.610807	-6.612523	-6.610741	0.000270	-1.969471	0.000270	-1.929031	-6.610807	-6.610807	5.449351e-09	-3.997041	5.449351e-09	-4.003670
N = 128	-6.610807	-6.611241	-6.610799	0.000067	-1.984670	0.000067	-1.964295	-6.610807	-6.610807	1.042794e-10	-3.999791	1.042794e-10	-5.591900
N = 256	-6.610807	-6.610916	-6.610806	0.000017	-1.992319	0.000017	-1.982094	-6.610807	-6.610807	2.507770e-09	-3.897269	2.507770e-09	-0.172920
N = 512	-6.610807	-6.610834	-6.610807	0.000004	-1.996154	0.000004	-1.991032	-6.610807	-6.610808	1.838673e-07	-3.969016	1.838673e-07	0.000192

Convergence of FEM Heat with x_bar = 0.637 and k_ratio = 4.0

	True Q	FDM 2 Q	FDM 2 Extrap Q	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 Q	FDM 4 Extrap Q	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	-8.118713	-8.460794	NaN	0.042135	NaN	NaN	NaN	-8.133066	NaN	1.767875e-03	NaN	NaN	NaN
N = 8	-8.118713	-8.208090	NaN	0.011009	-1.936363	NaN	NaN	-8.119747	NaN	1.273164e-04	-3.795526	NaN	NaN
N = 16	-8.118713	-8.141329	-8.117358	0.002953	-1.982599	0.002953	NaN	-8.118781	-8.118705	9.315394e-06	-3.945395	9.315394e-06	NaN
N = 32	-8.118713	-8.124385	-8.118622	0.000710	-1.995544	0.000710	-1.978241	-8.118718	-8.118713	5.385828e-07	-3.986111	5.385828e-07	-3.942616
N = 64	-8.118713	-8.120132	-8.118708	0.000175	-1.998879	0.000175	-1.994429	-8.118714	-8.118713	3.291068e-08	-3.996505	3.291068e-08	-3.986725
N = 128	-8.118713	-8.119068	-8.118713	0.000044	-1.999719	0.000044	-1.998599	-8.118713	-8.118713	1.880380e-09	-3.999265	1.880380e-09	-4.111819
N = 256	-8.118713	-8.118802	-8.118713	0.000011	-1.999929	0.000011	-1.999650	-8.118713	-8.118713	6.711957e-09	-3.957658	6.711957e-09	-0.361555
N = 512	-8.118713	-8.118736	-8.118713	0.000003	-1.999969	0.000003	-1.999914	-8.118713	-8.118713	3.259909e-08	-4.511233	3.259909e-08	-0.005554

Convergence of FEM Heat with x_bar = 0.637 and k_ratio = 4.0

	True Q	FEM 2 Q	FEM 2 Extrap Q	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 Q	FEM 4 Extrap Q	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	-8.118713	-8.531236	NaN	0.050811	NaN	NaN	NaN	-8.120861	NaN	2.645577e-04	NaN	NaN	NaN
N = 8	-8.118713	-8.240802	NaN	0.015038	-1.756541	NaN	NaN	-8.118855	NaN	1.741262e-05	-3.925377	NaN	NaN
N = 16	-8.118713	-8.152065	-8.113024	0.004812	-1.872107	0.004812	NaN	-8.118722	-8.118713	1.152191e-06	-3.979180	1.152191e-06	NaN
N = 32	-8.118713	-8.127440	-8.117983	0.001165	-1.934202	0.001165	-1.849443	-8.118714	-8.118713	7.010843e-08	-3.994197	7.010843e-08	-3.978225
N = 64	-8.118713	-8.120946	-8.118620	0.000287	-1.966597	0.000287	-1.922894	-8.118713	-8.118713	4.102870e-09	-3.998206	4.102870e-09	-4.072979
N = 128	-8.118713	-8.119278	-8.118702	0.000071	-1.983167	0.000071	-1.960943	-8.118713	-8.118713	3.686867e-09	-4.000771	3.686867e-09	3.285984
N = 256	-8.118713	-8.118855	-8.118713	0.000018	-1.991549	0.000018	-1.980339	-8.118713	-8.118713	3.290053e-08	-4.170779	3.290053e-08	-0.011173
N = 512	-8.118713	-8.118749	-8.118713	0.000004	-1.995766	0.000004	-1.990135	-8.118713	-8.118720	8.054074e-07	-5.371643	8.054074e-07	0.000028

Convergence of FDM Heat with x_bar = 0.637 and k_ratio = 8.0

	True Q	FDM 2 Q	FDM 2 Extrap Q	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 Q	FDM 4 Extrap Q	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	-9.254592	-9.698320	NaN	0.047947	NaN	NaN	NaN	-9.274971	NaN	2.202047e-03	NaN	NaN	NaN
N = 8	-9.254592	-9.370579	NaN	0.012533	-1.935704	NaN	NaN	-9.256061	NaN	1.587889e-04	-3.793663	NaN	NaN
N = 16	-9.254592	-9.283942	-9.252810	0.003365	-1.982521	0.003365	NaN	-9.254687	-9.254579	1.163493e-05	-3.944816	1.163493e-05	NaN
N = 32	-9.254592	-9.261952	-9.254472	0.000808	-1.995531	0.000808	-1.978140	-9.254598	-9.254591	6.722891e-07	-3.985957	6.722891e-07	-3.942006
N = 64	-9.254592	-9.256433	-9.254584	0.000200	-1.998876	0.000200	-1.994413	-9.254592	-9.254592	4.113100e-08	-3.996495	4.113100e-08	-3.984620
N = 128	-9.254592	-9.255052	-9.254591	0.000050	-1.999719	0.000050	-1.998595	-9.254592	-9.254592	2.593194e-09	-3.998491	2.593194e-09	-3.976249
N = 256	-9.254592	-9.254707	-9.254592	0.000012	-1.999930	0.000012	-1.999649	-9.254592	-9.254592	9.260652e-10	-4.020792	9.260652e-10	-0.664331
N = 512	-9.254592	-9.254620	-9.254592	0.000003	-1.999984	0.000003	-1.999911	-9.254592	-9.254591	7.338935e-08	-2.728397	7.338935e-08	-0.003551

Convergence of FEM Heat with x_bar = 0.637 and k_ratio = 8.0

	True Q	FEM 2 Q	FEM 2 Extrap Q	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 Q	FEM 4 Extrap Q	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	-9.254592	-9.779991	NaN	0.056772	NaN	NaN	NaN	-9.257242	NaN	2.863730e-04	NaN	NaN	NaN
N = 8	-9.254592	-9.409744	NaN	0.016765	-1.759727	NaN	NaN	-9.254767	NaN	1.891276e-05	-3.920464	NaN	NaN
N = 16	-9.254592	-9.297012	-9.247662	0.005337	-1.870848	0.005337	NaN	-9.254603	-9.254591	1.256283e-06	-3.978635	1.256283e-06	NaN
N = 32	-9.254592	-9.265704	-9.253665	0.001301	-1.932621	0.001301	-1.848271	-9.254592	-9.254592	7.620138e-08	-3.994470	7.620138e-08	-3.977814
N = 64	-9.254592	-9.257437	-9.254471	0.000321	-1.965528	0.000321	-1.921119	-9.254592	-9.254592	4.584467e-09	-3.998630	4.584467e-09	-4.034829
N = 128	-9.254592	-9.255312	-9.254576	0.000079	-1.982557	0.000079	-1.959714	-9.254592	-9.254592	3.057780e-10	-3.990863	3.057780e-10	-3.947723
N = 256	-9.254592	-9.254773	-9.254590	0.000020	-1.991225	0.000020	-1.979633	-9.254592	-9.254592	7.358738e-09	-4.171614	7.358738e-09	0.065911
N = 512	-9.254592	-9.254637	-9.254591	0.000005	-1.995597	0.000005	-1.989760	-9.254592	-9.254590	2.246012e-07	-2.898591	2.246012e-07	-0.000120

Convergence of FDM Heat with x_bar = 0.637 and k_ratio = 16.0

	True Q	FDM 2 Q	FDM 2 Extrap Q	FDM 2 % Error	FDM 2 Exact B	FDM 2 Extrap % Error	FDM 2 Extrap B	FDM 4 Q	FDM 4 Extrap Q	FDM 4 % Error	FDM 4 Exact B	FDM 4 Extrap % Error	FDM 4 Extrap B
N = 4	-9.980639	-10.508396	NaN	0.052878	NaN	NaN	NaN	-10.005565	NaN	2.497488e-03	NaN	NaN	NaN
N = 8	-9.980639	-10.118258	NaN	0.013789	-1.939192	NaN	NaN	-9.982436	NaN	1.800926e-04	-3.793667	NaN	NaN
N = 16	-9.980639	-10.015440	-9.978646	0.003687	-1.983481	0.003687	NaN	-9.980755	-9.980624	1.319577e-05	-3.944740	1.319577e-05	NaN
N = 32	-9.980639	-9.989364	-9.980505	0.000888	-1.995777	0.000888	-1.979342	-9.980646	-9.980638	7.625708e-07	-3.985932	7.625708e-07	-3.941925
N = 64	-9.980639	-9.982822	-9.980630	0.000220	-1.998938	0.000220	-1.994720	-9.980639	-9.980639	4.663319e-08	-3.996465	4.663319e-08	-3.985180
N = 128	-9.980639	-9.981185	-9.980638	0.000055	-1.999734	0.000055	-1.998673	-9.980639	-9.980639	3.043996e-09	-3.997001	3.043996e-09	-3.929875
N = 256	-9.980639	-9.980775	-9.980639	0.000014	-1.999931	0.000014	-1.999669	-9.980639	-9.980639	1.196629e-09	-4.015425	1.196629e-09	-0.345699
N = 512	-9.980639	-9.980673	-9.980639	0.000003	-1.999988	0.000003	-1.999911	-9.980639	-9.980638	8.575384e-08	-1.042866	8.575384e-08	-0.001550

Convergence of FEM Heat with x_bar = 0.637 and k_ratio = 16.0

	True Q	FEM 2 Q	FEM 2 Extrap Q	FEM 2 % Error	FEM 2 Exact B	FEM 2 Extrap % Error	FEM 2 Extrap B	FEM 4 Q	FEM 4 Extrap Q	FEM 4 % Error	FEM 4 Exact B	FEM 4 Extrap % Error	FEM 4 Extrap B
N = 4	-9.980639	-10.599453	NaN	0.062001	NaN	NaN	NaN	-9.983783	NaN	3.150905e-04	NaN	NaN	NaN
N = 8	-9.980639	-10.162539	NaN	0.018225	-1.766361	NaN	NaN	-9.980847	NaN	2.083072e-05	-3.918981	NaN	NaN
N = 16	-9.980639	-10.030332	-9.972971	0.005752	-1.872008	0.005752	NaN	-9.980652	-9.980638	1.385283e-06	-3.978393	1.385283e-06	NaN
N = 32	-9.980639	-9.993657	-9.979577	0.001411	-1.932506	0.001411	-1.849908	-9.980639	-9.980639	8.398012e-08	-3.994484	8.398012e-08	-3.977219
N = 64	-9.980639	-9.983973	-9.980498	0.000348	-1.965272	0.000348	-1.921052	-9.980639	-9.980639	5.199889e-09	-3.998709	5.199889e-09	-3.995296
N = 128	-9.980639	-9.981482	-9.980620	0.000086	-1.982375	0.000086	-1.959430	-9.980639	-9.980639	9.369049e-10	-4.010246	9.369049e-10	-2.067819
N = 256	-9.980639	-9.980851	-9.980636	0.000021	-1.991121	0.000021	-1.979424	-9.980639	-9.980638	2.628290e-08	-4.208162	2.628290e-08	-0.016616
N = 512	-9.980639	-9.980692	-9.980638	0.000005	-1.995533	0.000005	-1.989640	-9.980639	-9.980633	5.895397e-07	-3.717511	5.895397e-07	-0.000039

6 Python Code

6.1 main.py

```
# main.py

import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import math
import sys

import case1
import Bar
from case1 import solve, get_analytical_datapoints, get_analytical_heat_convection
from common import calc_extrapolated, calc_error, calc_beta, fdm_heat_extraction,
    fem_heat_extraction, heat_extraction


bar1 = Bar.Bar(
    radius=0.1,
    k=0.5,
    h=0.25
)
bar2 = Bar.Bar(
    radius=0.1,
    k=2,
    h=0.25
)

data_dict = {}
data_dict["k_ratios"] = np.array([1/16, 1/8, 1/4, 1/2, 1, 2, 4, 8, 16])
data_dict["x_bar_values"] = np.array([1/2, 2/np.pi])
data_dict["length"] = 1

def analytical():
    '''Analytical solutions with varying k_ratio'''
    print("Section 1: Analytical")
    file_path = "images/analytical/"

    l = data_dict["length"]

    # Calculate the results for every k ratio for x_bar_1
    x_bar_1 = data_dict["x_bar_values"][0]

    data_dict["analytical"] = {}
    data_dict["analytical"]["x_bar_1"] = {}
    data_dict["analytical"]["x_bar_1"]["x_values_fine"] = np.linspace(0, l, 50)
    data_dict["analytical"]["x_bar_1"]["x_values_fine"] =
        np.sort(np.append(data_dict["analytical"]["x_bar_1"]["x_values_fine"], x_bar_1))
```

```
data_dict["analytical"]["x_bar_1"]["x_values_course"] = np.linspace(0, 1, 9)
data_dict["analytical"]["x_bar_1"]["x_values_course"] =
    np.sort(np.append(data_dict["analytical"]["x_bar_1"]["x_values_course"], x_bar_1))

results = []
results_course = []
for k_ratio in data_dict["k_ratios"]:
    Bar.set_k_ratio(k_ratio, bar1, bar2) # Set the k value of bar2 = k*bar1
    result = get_analytical_datapoints(data_dict["analytical"]["x_bar_1"]["x_values_fine"],
                                       bar1=bar1,
                                       bar2=bar2,
                                       x_bar=x_bar_1,
                                       l=l)
    result_course =
        get_analytical_datapoints(data_dict["analytical"]["x_bar_1"]["x_values_course"],
                                       bar1=bar1,
                                       bar2=bar2,
                                       x_bar=x_bar_1,
                                       l=l)
    results.append(result)
    results_course.append(result_course)

data_dict["analytical"]["x_bar_1"]["temp_results"] = np.array(results)

df_x_bar_1 = pd.DataFrame(results_course, index=[f'k2/k1 = {a}' for a in data_dict["k_ratios"]],
    columns=[f'x = {x:.3f}' for x in data_dict["analytical"]["x_bar_1"]["x_values_course"]])

print("Analytical Temperature Results x_bar = 0.5:")
print(df_x_bar_1.to_string())
print("\n" * 2)

# Plotting x_bar_1
plt.figure(figsize=(10, 6))
for idx, k_ratio in enumerate(data_dict["k_ratios"]):
    plt.plot(data_dict["analytical"]["x_bar_1"]["x_values_fine"],
        data_dict["analytical"]["x_bar_1"]["temp_results"][idx], label=f'k2/k1 = {k_ratio}')
plt.axvline(x=x_bar_1, color='r', linestyle='--', label='x_bar')
plt.xlabel('Position (cm)')
plt.ylabel('Temperature (C)')
plt.legend()
plt.grid(True)
plt.savefig(f'{file_path}x_bar_1/temperature_distribution.png', dpi=300)
#plt.show()
plt.clf()

# Calculate the temperature results for every k ratio for x_bar_1
x_bar_2 = data_dict["x_bar_values"][1]

data_dict["analytical"]["x_bar_2"] = {}
data_dict["analytical"]["x_bar_2"]["x_values_fine"] = np.linspace(0, 1, 50)
```

```
data_dict["analytical"]["x_bar_2"]["x_values_fine"] =
    np.sort(np.append(data_dict["analytical"]["x_bar_2"]["x_values_fine"], x_bar_2))
data_dict["analytical"]["x_bar_2"]["x_values_course"] = np.linspace(0, l, 9)
data_dict["analytical"]["x_bar_2"]["x_values_course"] =
    np.sort(np.append(data_dict["analytical"]["x_bar_2"]["x_values_course"], x_bar_2))

results = []
results_course = []
for k_ratio in data_dict["k_ratios"]:
    Bar.set_k_ratio(k_ratio, bar1, bar2)
    result = get_analytical_datapoints(data_dict["analytical"]["x_bar_2"]["x_values_fine"],
                                       bar1=bar1,
                                       bar2=bar2,
                                       x_bar=x_bar_2,
                                       l=l)
    result_course =
        get_analytical_datapoints(data_dict["analytical"]["x_bar_2"]["x_values_course"],
                                       bar1=bar1,
                                       bar2=bar2,
                                       x_bar=x_bar_2,
                                       l=l)
    results.append(result)
    results_course.append(result_course)

data_dict["analytical"]["x_bar_2"]["temp_results"] = np.array(results)

df_x_bar_2 = pd.DataFrame(results_course, index=[f'k2/k1 = {a}' for a in data_dict["k_ratios"]],
    columns=[f'x = {x:.3f}' for x in data_dict["analytical"]["x_bar_2"]["x_values_course"]])

print("Analytical Temperature Results x_bar = 2/pi:")
print(df_x_bar_2.to_string())
print("\n" * 2)

# Plotting x_bar_2
plt.figure(figsize=(10, 6))
for idx, k_ratio in enumerate(data_dict["k_ratios"]):
    plt.plot(data_dict["analytical"]["x_bar_2"]["x_values_fine"],
        data_dict["analytical"]["x_bar_2"]["temp_results"][idx], label=f'k2/k1 = {k_ratio}')
plt.axvline(x=x_bar_2, color='r', linestyle='--', label='x_bar')
plt.xlabel('Position (cm)')
plt.ylabel('Temperature (C)')
plt.legend()
plt.grid(True)
plt.savefig(f'{file_path}x_bar_2/temperature_distribution.png', dpi=300)
#plt.show()
plt.clf()



# Calculate the analytical heat convection leaving the rod at x=l for varying k ratio
```

```
results_1 = []
results_2 = []
for k_ratio in data_dict["k_ratios"]:
    Bar.set_k_ratio(k_ratio, bar1, bar2)
    result_1 = get_analytical_heat_convection(x=l,
                                              bar1=bar1,
                                              bar2=bar2,
                                              x_bar=x_bar_1,
                                              l=l)
    result_2 = get_analytical_heat_convection(x=l,
                                              bar1=bar1,
                                              bar2=bar2,
                                              x_bar=x_bar_2,
                                              l=l)
    results_1.append([result_1])
    results_2.append([result_2])

data_dict["analytical"]["x_bar_1"]["heat_results"] = np.array(results_1)
data_dict["analytical"]["x_bar_2"]["heat_results"] = np.array(results_2)

df_x_bar_1 = pd.DataFrame(results_1, index=[f'k2/k1 = {a}' for a in data_dict["k_ratios"]],
    columns=["Heat Convection"])
df_x_bar_2 = pd.DataFrame(results_2, index=[f'k2/k1 = {a}' for a in data_dict["k_ratios"]],
    columns=["Heat Convection"])

print("Analytical Heat Convection Results x_bar = 0.5:")
print(df_x_bar_1.to_string())
print("\n" * 2)

print("Analytical Heat Convection Results x_bar = 2/pi:")
print(df_x_bar_2.to_string())
print("\n" * 2)

# Plotting x_bar_1 heat convection
plt.figure(figsize=(10, 6))
plt.plot(data_dict["k_ratios"], data_dict["analytical"]["x_bar_1"]["heat_results"], label=f'Heat
    Convection')
plt.xlabel('k2/k1')
plt.ylabel('Q_Dot (W)')
plt.legend()
plt.grid(True)
plt.savefig(f'{file_path}x_bar_1/heat_convection.png', dpi=300)
#plt.show()
plt.clf()

# Plotting x_bar_2 heat convection
plt.figure(figsize=(10, 6))
plt.plot(data_dict["k_ratios"], data_dict["analytical"]["x_bar_2"]["heat_results"], label=f'Heat
    Convection')
```

```
    plt.xlabel('k2/k1')
    plt.ylabel('Q_Dot (W)')
    plt.legend()
    plt.grid(True)
    plt.savefig(f'{file_path}x_bar_2/heat_convection.png', dpi=300)
    #plt.show()
    plt.clf()


def section_2(method="FDM"):
    '''FDM and FEM with varying k_ratio'''
    print(f"Section 2: {method} with Varying k_ratio")
    file_path = f"images/{method}/"

    l = data_dict["length"]

    data_dict["section2"] = {}
    data_dict["section2"]["num_sections"] = 8


    # Calculate the results for every k ratio for x_bar_1
    x_bar_1 = data_dict["x_bar_values"][0]

    # For every k_ratio, calculate the FDM and FEM temperature results
    fdm_results = []
    fem_results = []
    x_fdm = []
    x_fem = []
    for k_ratio in data_dict["k_ratios"]:
        Bar.set_k_ratio(k_ratio, bar1, bar2) # Set the k value of bar2 = k*bar1
        fdm_temps, x_fdm = solve(bar1=bar1,
                                 bar2=bar2,
                                 num_sections=data_dict["section2"]["num_sections"],
                                 x_bar=x_bar_1,
                                 l=l,
                                 method=method,
                                 order=2)
        fem_temps, x_fem = solve(bar1=bar1,
                                 bar2=bar2,
                                 num_sections=data_dict["section2"]["num_sections"],
                                 x_bar=x_bar_1,
                                 l=l,
                                 method=method,
                                 order=4)

        fdm_results.append(fdm_temps)
        fem_results.append(fem_temps)

    data_dict["section2"]["x_bar_1"] = {}
    data_dict["section2"]["x_bar_1"]["x_values"] = x_fdm # fdm and fem use same x_values
```

```
data_dict["section2"]["x_bar_1"]["fdm_results"] = np.array(fdm_results)
data_dict["section2"]["x_bar_1"]["fem_results"] = np.array(fem_results)

df_fdm = pd.DataFrame(fdm_results, index=[f'k2/k1 = {a}' for a in data_dict["k_ratios"]],
    columns=[f'x = {x:.3f}' for x in x_fdm])
df_fem = pd.DataFrame(fem_results, index=[f'k2/k1 = {a}' for a in data_dict["k_ratios"]],
    columns=[f'x = {x:.3f}' for x in x_fem])


foo = "2nd Order" if method=="FDM" else "p=1"
bar = "4th Order" if method=="FDM" else "p=2"

print(f"{method} {foo} Temperature Results x_bar = 0.5:")
print(df_fdm.to_string())
print("\n" * 2)
print(f"{method} {bar} Temperature Results x_bar = 0.5:")
print(df_fem.to_string())
print("\n" * 2)

# Now that the data is gathered for FDM and FEM, plot it
# Plotting x_bar_1, FDM
plt.figure(figsize=(10, 6))
for idx, k_ratio in enumerate(data_dict["k_ratios"]):
    plt.plot(data_dict["section2"]["x_bar_1"]["x_values"],
        data_dict["section2"]["x_bar_1"]["fdm_results"][idx], label=f'k2/k1 = {k_ratio}')
plt.axvline(x=x_bar_1, color='r', linestyle='--', label='x_bar')
plt.xlabel('Position (cm)')
plt.ylabel('Temperature (C)')
plt.legend()
plt.grid(True)
plt.savefig(f'{file_path}x_bar_1/temperature_distribution_1.png', dpi=300)
#plt.show()
plt.clf()

# Plotting x_bar_1
plt.figure(figsize=(10, 6))
for idx, k_ratio in enumerate(data_dict["k_ratios"]):
    plt.plot(data_dict["section2"]["x_bar_1"]["x_values"],
        data_dict["section2"]["x_bar_1"]["fem_results"][idx], label=f'k2/k1 = {k_ratio}')
plt.axvline(x=x_bar_1, color='r', linestyle='--', label='x_bar')
plt.xlabel('Position (cm)')
plt.ylabel('Temperature (C)')
plt.legend()
plt.grid(True)
plt.savefig(f'{file_path}x_bar_1/temperature_distribution_2.png', dpi=300)
#plt.show()
plt.clf()


# Calculate the results for every k ratio for x_bar_2
```

```
x_bar_2 = data_dict["x_bar_values"][1]

# For every k_ratio, calculate the FDM and FEM temperature results
fdm_results = []
fem_results = []
x_fdm = []
x_fem = []
for k_ratio in data_dict["k_ratios"]:
    Bar.set_k_ratio(k_ratio, bar1, bar2) # Set the k value of bar2 = k*bar1
    fdm_temps, x_fdm = solve(bar1=bar1,
                             bar2=bar2,
                             num_sections=data_dict["section2"]["num_sections"],
                             x_bar=x_bar_2,
                             l=l,
                             method=method, order=2)
    fem_temps, x_fem = solve(bar1=bar1,
                             bar2=bar2,
                             num_sections=data_dict["section2"]["num_sections"],
                             x_bar=x_bar_2,
                             l=l,
                             method=method, order=4)

    fdm_results.append(fdm_temps)
    fem_results.append(fem_temps)

data_dict["section2"]["x_bar_2"] = {}
data_dict["section2"]["x_bar_2"]["x_values"] = x_fdm # fdm and fem use same x_values
data_dict["section2"]["x_bar_2"]["fdm_results"] = np.array(fdm_results)
data_dict["section2"]["x_bar_2"]["fem_results"] = np.array(fem_results)

df_fdm = pd.DataFrame(fdm_results, index=[f'k2/k1 = {a}' for a in data_dict["k_ratios"]],
    columns=[f'x = {x:.3f}' for x in x_fdm])
df_fem = pd.DataFrame(fem_results, index=[f'k2/k1 = {a}' for a in data_dict["k_ratios"]],
    columns=[f'x = {x:.3f}' for x in x_fem])

print(f"{method} {foo} Temperature Results x_bar = 2/pi:")
print(df_fdm.to_string())
print("\n" * 2)
print(f"{method} {bar} Temperature Results x_bar = 2/pi:")
print(df_fem.to_string())
print("\n" * 2)

# Plotting x_bar_2, FDM
plt.figure(figsize=(10, 6))
for idx, k_ratio in enumerate(data_dict["k_ratios"]):
    plt.plot(data_dict["section2"]["x_bar_2"]["x_values"],
        data_dict["section2"]["x_bar_2"]["fdm_results"][idx], label=f'k2/k1 = {k_ratio}')
plt.axvline(x=x_bar_2, color='r', linestyle='--', label='x_bar')
plt.xlabel('Position (cm)')
plt.ylabel('Temperature (C)')
```

```
plt.legend()
plt.grid(True)
plt.savefig(f'{file_path}x_bar_2/temperature_distribution_1.png', dpi=300)
#plt.show()
plt.clf()

# Plotting x_bar_2, FEM
plt.figure(figsize=(10, 6))
for idx, k_ratio in enumerate(data_dict["k_ratios"]):
    plt.plot(data_dict["section2"]["x_bar_2"]["x_values"],
        data_dict["section2"]["x_bar_2"]["fem_results"][idx], label=f'k2/k1 = {k_ratio}')
plt.axvline(x=x_bar_2, color='r', linestyle='--', label='x_bar')
plt.xlabel('Position (cm)')
plt.ylabel('Temperature (C)')
plt.legend()
plt.grid(True)
plt.savefig(f'{file_path}x_bar_2/temperature_distribution_2.png', dpi=300)
#plt.show()
plt.clf()



# After calculating temperature values, extract the heat convection at x=l

# x_bar_1
# for every k ratio, calculate the heat convection from fdm and fem temp results
fdm_heat_results = []
fem_heat_results = []
for i, k_ratio in enumerate(data_dict["k_ratios"]):
    Bar.set_k_ratio(k_ratio, bar1, bar2) # Set the k value of bar2 = k*bar1
    dx = data_dict["section2"]["x_bar_1"]["x_values"][-1] -
        data_dict["section2"]["x_bar_1"]["x_values"][-2]
    (fdm_t0, fdm_t1) = data_dict["section2"]["x_bar_1"]["fdm_results"][i][-2:]
    fdm_heat_result = heat_extraction(fdm_t0, fdm_t1, dx, bar2, order=2, method=method)
    fdm_heat_results.append(fdm_heat_result)

    (fem_t0, fem_t1) = data_dict["section2"]["x_bar_1"]["fem_results"][i][-2:]
    fem_heat_result = heat_extraction(fem_t0, fem_t1, dx, bar2, order=2, method=method)
    fem_heat_results.append(fem_heat_result)


data_dict["section2"]["x_bar_1"]["fdm_heat_results"] = np.array(fdm_heat_results)
data_dict["section2"]["x_bar_1"]["fem_heat_results"] = np.array(fem_heat_results)

# Create DataFrame with two columns for FDM and FEM heat results
df_heat = pd.DataFrame(
    {
        f"{foo} Heat Convection": fdm_heat_results,
        f"{bar} Heat Convection": fem_heat_results
    },
```

```
    index=[f'k2/k1 = {a}' for a in data_dict["k_ratios"]]
)

# Print the DataFrame with labeled columns and indices
print(f"{method} Heat Convection Results for x_bar = 0.5:")
print(df_heat.to_string())
print("\n")

# x_bar_2
# for every k ratio, calculate the heat convection from fdm and fem temp results
fdm_heat_results = []
fem_heat_results = []
for i, k_ratio in enumerate(data_dict["k_ratios"]):
    Bar.set_k_ratio(k_ratio, bar1, bar2) # Set the k value of bar2 = k*bar1
    dx = data_dict["section2"]["x_bar_2"]["x_values"][-1] -
        data_dict["section2"]["x_bar_2"]["x_values"][-2]
    (fdm_t0, fdm_t1) = data_dict["section2"]["x_bar_2"]["fdm_results"][i][-2:]
    fdm_heat_result = heat_extraction(fdm_t0, fdm_t1, dx, bar2, order=2, method=method)
    fdm_heat_results.append(fdm_heat_result)

    (fem_t0, fem_t1) = data_dict["section2"]["x_bar_2"]["fem_results"][i][-2:]
    fem_heat_result = heat_extraction(fem_t0, fem_t1, dx, bar2, order=4, method=method)
    fem_heat_results.append(fem_heat_result)


data_dict["section2"]["x_bar_2"]["fdm_heat_results"] = np.array(fdm_heat_results)
data_dict["section2"]["x_bar_2"]["fem_heat_results"] = np.array(fem_heat_results)

# Create DataFrame with two columns for FDM and FEM heat results
df_heat = pd.DataFrame(
    {
        f"{foo} Heat Convection": fdm_heat_results,
        f"{bar} Heat Convection": fem_heat_results
    },
    index=[f'k2/k1 = {a}' for a in data_dict["k_ratios"]]
)

# Print the DataFrame with labeled columns and indices
print(f"{method} Heat Convection Results for x_bar = 2/pi:")
print(df_heat.to_string())
print("\n")

# Plotting x_bar_1 heat convection
plt.figure(figsize=(10, 6))
plt.plot(data_dict["k_ratios"], data_dict["section2"]["x_bar_1"]["fdm_heat_results"], label=f'2nd
    Order')
plt.plot(data_dict["k_ratios"], data_dict["section2"]["x_bar_1"]["fem_heat_results"], label=f'4th
    Order')
#plt.plot(data_dict["k_ratios"], data_dict["analytical"]["x_bar_1"]["heat_results"],
    label=f'Analytical Heat Convection')
```

```
    plt.xlabel('k2/k1')
    plt.ylabel('Q_Dot (W)')
    plt.legend()
    plt.grid(True)
    plt.savefig(f'{file_path}x_bar_1/heat_convection.png', dpi=300)
    #plt.show()
    plt.clf()

    # Plotting x_bar_2 heat convection
    plt.figure(figsize=(10, 6))
    plt.plot(data_dict["k_ratios"], data_dict["section2"]["x_bar_2"]["fdm_heat_results"], label=f'2nd
        Order')
    plt.plot(data_dict["k_ratios"], data_dict["section2"]["x_bar_2"]["fem_heat_results"], label=f'4th
        Order')
    #plt.plot(data_dict["k_ratios"], data_dict["analytical"]["x_bar_2"]["heat_results"],
        label=f'Analytical Heat Convection')
    plt.xlabel('k2/k1')
    plt.ylabel('Q_Dot (W)')
    plt.legend()
    plt.grid(True)
    plt.savefig(f'{file_path}x_bar_2/heat_convection.png', dpi=300)
    #plt.show()
    plt.clf()


def temp_convergence():
    '''Convergence of FDM and FEM of interface temperature'''
    print("Section 3 Temp: Convergence of FDM and FEM Temp with Varying k Ratio")
    file_path = "images/convergence/"
    l = data_dict["length"]

    data_dict["convergence"] = {}
    data_dict["convergence"]["num_sections"] = [4, 8, 16, 32, 64, 128, 256, 512]

    for x_bar_idx, x_bar in enumerate(data_dict["x_bar_values"]):
        num_k_ratios = len(data_dict["k_ratios"])
        num_rows = math.ceil(num_k_ratios / 2)

        fig_temp, axs_temp = plt.subplots(num_rows, 2, figsize=(15, 5 * num_rows))

        axs_temp = axs_temp.flatten()

        for idx, k_ratio in enumerate(data_dict["k_ratios"]):
            Bar.set_k_ratio(k_ratio, bar1, bar2)

            # Analytical temperature at the interface
            analy_interface_temp = get_analytical_datapoints([x_bar], bar1, bar2, x_bar, l)[0]
            convergence_data = {
                "fdm_2_values": [], "fdm_4_values": [], "fem_2_values": [], "fem_4_values": [],
                "fdm_2_errors": [], "fdm_4_errors": [], "fem_2_errors": [], "fem_4_errors": [],
```

```
    "fdm_2_extrap": [], "fdm_4_extrap": [], "fem_2_extrap": [], "fem_4_extrap": [],
    "fdm_2_b": [], "fdm_4_b": [], "fem_2_b": [], "fem_4_b": []
}

# Calculate temperature values and errors for different section counts
for num_sections in data_dict["convergence"]["num_sections"]:
    temp_results = {
        "FDM_2": solve(bar1, bar2, num_sections, x_bar, l, method="FDM", order=2)[0],
        "FDM_4": solve(bar1, bar2, num_sections, x_bar, l, method="FDM", order=4)[0],
        "FEM_2": solve(bar1, bar2, num_sections, x_bar, l, method="FEM", order=2)[0],
        "FEM_4": solve(bar1, bar2, num_sections, x_bar, l, method="FEM", order=4)[0]
    }

    interface_idx = num_sections // 2
    for method, temps in temp_results.items():
        interface_temp = temps[interface_idx]
        convergence_data[f"{method.lower()}_values"].append(interface_temp)
        error = calc_error(analy_interface_temp, interface_temp)
        convergence_data[f"{method.lower()}_errors"].append(error)

# Calculate convergence rates ‘B‘ for each method and apply extrapolation from the third
    entry onward
for method in ["fdm_2", "fdm_4", "fem_2", "fem_4"]:
    values = convergence_data[f"{method}_values"]
    errors = convergence_data[f"{method}_errors"]

    # Initialize ‘extrap‘ with NaNs for the first two entries
    extrapolated_values = [float('NaN')] * 2
    b_values = [float('NaN')] # Initialize ‘B‘ list with NaN for the first entry

    # Compute extrapolated values starting from the third entry if possible
    for i in range(2, len(values)):
        if i >= 2:
            extrap_value = calc_extrapolated(values[i - 2], values[i - 1], values[i])
            extrapolated_values.append(extrap_value)

            # Calculate extrapolated error for the current value
            extrap_error = calc_error(extrap_value, values[i])
            errors[i] = extrap_error
        else:
            extrapolated_values.append(float('NaN'))

    # Calculate convergence rates ‘B‘ for each consecutive pair
    for i in range(1, len(values)):
        beta = calc_beta(analy_interface_temp, values[i], values[i - 1],
                         data_dict["convergence"]["num_sections"][i],
                         data_dict["convergence"]["num_sections"][i - 1])
        b_values.append(beta)

    # Store the calculated ‘B‘ and extrapolated values in the convergence data
```

```
    convergence_data[f"{method}_b"] = b_values
    convergence_data[f"{method}_extrap"] = extrapolated_values



# Prepare data for FDM and FEM tables
rows = data_dict["convergence"]["num_sections"]
fdm_table_data = []
fem_table_data = []

for i, num_sections in enumerate(rows):
    # Append data for FDM
    fdm_table_data.append([
        analy_interface_temp,
        convergence_data["fdm_2_values"][i],
        convergence_data["fdm_2_extrap"][i],
        convergence_data["fdm_2_errors"][i],
        convergence_data["fdm_2_b"][i],
        calc_error(convergence_data["fdm_2_extrap"][i],
            convergence_data["fdm_2_values"][i]) if i >= 2 else float('NaN'),
        calc_beta(convergence_data["fdm_2_extrap"][i], convergence_data["fdm_2_values"][i],
            convergence_data["fdm_2_values"][i-1], num_sections, rows[i-1]) if i >= 3 else
            float('NaN'),

        convergence_data["fdm_4_values"][i],
        convergence_data["fdm_4_extrap"][i],
        convergence_data["fdm_4_errors"][i],
        convergence_data["fdm_4_b"][i],
        calc_error(convergence_data["fdm_4_extrap"][i],
            convergence_data["fdm_4_values"][i]) if i >= 2 else float('NaN'),
        calc_beta(convergence_data["fdm_4_extrap"][i], convergence_data["fdm_4_values"][i],
            convergence_data["fdm_4_values"][i-1], num_sections, rows[i-1]) if i >= 3 else
            float('NaN'),
    ])

    # Append data for FEM
    fem_table_data.append([
        analy_interface_temp,
        convergence_data["fem_2_values"][i],
        convergence_data["fem_2_extrap"][i],
        convergence_data["fem_2_errors"][i],
        convergence_data["fem_2_b"][i],
        calc_error(convergence_data["fem_2_extrap"][i],
            convergence_data["fem_2_values"][i]) if i >= 2 else float('NaN'),
        calc_beta(convergence_data["fem_2_extrap"][i], convergence_data["fem_2_values"][i],
            convergence_data["fem_2_values"][i-1], num_sections, rows[i-1]) if i >= 3 else
            float('NaN'),

        convergence_data["fem_4_values"][i],
        convergence_data["fem_4_extrap"][i],
```

```
            convergence_data["fem_4_errors"][i],
            convergence_data["fem_4_b"][i],
            calc_error(convergence_data["fem_4_extrap"][i],
                convergence_data["fem_4_values"][i]) if i >= 2 else float('NaN'),
            calc_beta(convergence_data["fem_4_extrap"][i], convergence_data["fem_4_values"][i],
                convergence_data["fem_4_values"][i-1], num_sections, rows[i-1]) if i >= 3 else
                float('NaN'),
        ])

        # Store extrapolated % errors for plotting
        if i >= 2:
            if "fdm_2_extrap_errors" not in convergence_data:
                convergence_data["fdm_2_extrap_errors"] = [float('NaN')] * 2
                convergence_data["fdm_4_extrap_errors"] = [float('NaN')] * 2
                convergence_data["fem_2_extrap_errors"] = [float('NaN')] * 2
                convergence_data["fem_4_extrap_errors"] = [float('NaN')] * 2
            convergence_data["fdm_2_extrap_errors"].append(calc_error(convergence_data["fdm_2_extrap"]
                convergence_data["fdm_2_values"][i]))
            convergence_data["fdm_4_extrap_errors"].append(calc_error(convergence_data["fdm_4_extrap"]
                convergence_data["fdm_4_values"][i]))
            convergence_data["fem_2_extrap_errors"].append(calc_error(convergence_data["fem_2_extrap"]
                convergence_data["fem_2_values"][i]))
            convergence_data["fem_4_extrap_errors"].append(calc_error(convergence_data["fem_4_extrap"]
                convergence_data["fem_4_values"][i]))
        else:
            if "fdm_2_extrap_errors" in convergence_data:
                convergence_data["fdm_2_extrap_errors"].append(float('NaN'))
                convergence_data["fdm_4_extrap_errors"].append(float('NaN'))
                convergence_data["fem_2_extrap_errors"].append(float('NaN'))
                convergence_data["fem_4_extrap_errors"].append(float('NaN'))


# Update columns for new fields
fdm_columns = [
    'True T', 'FDM 2 T', 'FDM 2 Extrap T', 'FDM 2 % Error', 'FDM 2 Exact B',
    'FDM 2 Extrap % Error', 'FDM 2 Extrap B',
    'FDM 4 T', 'FDM 4 Extrap T', 'FDM 4 % Error', 'FDM 4 Exact B',
    'FDM 4 Extrap % Error', 'FDM 4 Extrap B'
]

fem_columns = [
    'True T', 'FEM 2 T', 'FEM 2 Extrap T', 'FEM 2 % Error', 'FEM 2 Exact B',
    'FEM 2 Extrap % Error', 'FEM 2 Extrap B',
    'FEM 4 T', 'FEM 4 Extrap T', 'FEM 4 % Error', 'FEM 4 Exact B',
    'FEM 4 Extrap % Error', 'FEM 4 Extrap B'
]

fdm_df = pd.DataFrame(fdm_table_data, index=[f'N = {n}' for n in rows],
    columns=fdm_columns)
```

```
    fem_df = pd.DataFrame(fem_table_data, index=[f'N = {n}' for n in rows],
        columns=fem_columns)

    print(f"Convergence of FDM Temperature at x_bar = {x_bar:.3f} and k_ratio = {k_ratio}")
    print(fdm_df.to_string())
    print("\n")

    print(f"Convergence of FEM Temperature at x_bar = {x_bar:.3f} and k_ratio = {k_ratio}")
    print(fem_df.to_string())
    print("\n")

    # Plot convergence for FDM and FEM errors and extrapolated errors
    ax_temp = axs_temp[idx]
    num_sections = data_dict["convergence"]["num_sections"]

    # Plot analytical error convergence for FDM and FEM, orders 2 and 4
    ax_temp.plot(num_sections, convergence_data["fdm_2_errors"], 'o-', label="FDM 2 % Error
        (Analytical)")
    ax_temp.plot(num_sections, convergence_data["fdm_4_errors"], 's-', label="FDM 4 % Error
        (Analytical)")
    ax_temp.plot(num_sections, convergence_data["fem_2_errors"], '^-', label="FEM 2 % Error
        (Analytical)")
    ax_temp.plot(num_sections, convergence_data["fem_4_errors"], 'd-', label="FEM 4 % Error
        (Analytical)")

    # Plot extrapolated error convergence for FDM and FEM, orders 2 and 4
    ax_temp.plot(num_sections, convergence_data["fdm_2_extrap_errors"], 'o--', label="FDM 2 %
        Error (Extrapolated)")
    ax_temp.plot(num_sections, convergence_data["fdm_4_extrap_errors"], 's--', label="FDM 4 %
        Error (Extrapolated)")
    ax_temp.plot(num_sections, convergence_data["fem_2_extrap_errors"], '^--', label="FEM 2 %
        Error (Extrapolated)")
    ax_temp.plot(num_sections, convergence_data["fem_4_extrap_errors"], 'd--', label="FEM 4 %
        Error (Extrapolated)")

    # Set log scales, labels, and legends
    ax_temp.set_xscale("log")
    ax_temp.set_yscale("log")
    ax_temp.set_xlabel("Number of Sections (N)")
    ax_temp.set_ylabel("% Error")
    ax_temp.set_title(f"Interface Temp Convergence (k_ratio={k_ratio})")
    ax_temp.legend()
    ax_temp.grid(True)


fig_temp.tight_layout(rect=[0, 0.03, 1, 0.95])
x_bar_path = "x_bar_1/" if x_bar_idx == 0 else "x_bar_2/"
fig_temp.savefig(f"{file_path+x_bar_path}temp_convergence.png", dpi=300)
```

```
def heat_convergence():
    '''Convergence of FDM and FEM of interface heat convection'''
    print("Section 3 Temp: Convergence of FDM and FEM Temp with Varying k Ratio")
    file_path = "images/convergence/"
    l = data_dict["length"]

    data_dict["convergence"] = {}
    data_dict["convergence"]["num_sections"] = [4, 8, 16, 32, 64, 128, 256, 512]

    for x_bar_idx, x_bar in enumerate(data_dict["x_bar_values"]):
        num_k_ratios = len(data_dict["k_ratios"])
        num_rows = math.ceil(num_k_ratios / 2)

        fig_heat, axs_heat = plt.subplots(num_rows, 2, figsize=(15, 5 * num_rows))

        axs_heat = axs_heat.flatten()

        for idx, k_ratio in enumerate(data_dict["k_ratios"]):
            Bar.set_k_ratio(k_ratio, bar1, bar2)

            # Analytical temperature at the interface
            analy_heat = get_analytical_heat_convection(l, bar1, bar2, x_bar, l)
            convergence_data = {
                "fdm_2_values": [], "fdm_4_values": [], "fem_2_values": [], "fem_4_values": [],
                "fdm_2_errors": [], "fdm_4_errors": [], "fem_2_errors": [], "fem_4_errors": [],
                "fdm_2_extrap": [], "fdm_4_extrap": [], "fem_2_extrap": [], "fem_4_extrap": [],
                "fdm_2_b": [], "fdm_4_b": [], "fem_2_b": [], "fem_4_b": []
            }

            # Calculate temperature values and errors for different section counts
            for num_sections in data_dict["convergence"]["num_sections"]:
                temp_results = {
                    "FDM_2": solve(bar1, bar2, num_sections, x_bar, l, method="FDM", order=2)[0],
                    "FDM_4": solve(bar1, bar2, num_sections, x_bar, l, method="FDM", order=4)[0],
                    "FEM_2": solve(bar1, bar2, num_sections, x_bar, l, method="FEM", order=2)[0],
                    "FEM_4": solve(bar1, bar2, num_sections, x_bar, l, method="FEM", order=4)[0]
                }

                dx = (l - x_bar) / (num_sections // 2)
                for method, temps in temp_results.items():
                    heat = heat_extraction(temps[-2], temps[-1], dx, bar2, order=int(method[-1]),
                        method=method[:3])
                    convergence_data[f"{method.lower()}_values"].append(heat)
                    error = calc_error(analy_heat, heat)
                    convergence_data[f"{method.lower()}_errors"].append(error)

            # Calculate convergence rates `B` for each method and apply extrapolation from the third
                entry onward
            for method in ["fdm_2", "fdm_4", "fem_2", "fem_4"]:
                values = convergence_data[f"{method}_values"]
```

```
    errors = convergence_data[f"{method}_errors"]

    # Initialize ‘extrap‘ with NaNs for the first two entries
    extrapolated_values = [float('NaN')] * 2
    b_values = [float('NaN')] # Initialize ‘B‘ list with NaN for the first entry

    # Compute extrapolated values starting from the third entry if possible
    for i in range(2, len(values)):
        if i >= 2:
            extrap_value = calc_extrapolated(values[i - 2], values[i - 1], values[i])
            extrapolated_values.append(extrap_value)

            # Calculate extrapolated error for the current value
            extrap_error = calc_error(extrap_value, values[i])
            errors[i] = extrap_error
        else:
            extrapolated_values.append(float('NaN'))

    # Calculate convergence rates ‘B‘ for each consecutive pair
    for i in range(1, len(values)):
        beta = calc_beta(analy_heat, values[i], values[i - 1],
                         data_dict["convergence"]["num_sections"][i],
                         data_dict["convergence"]["num_sections"][i - 1])
        b_values.append(beta)

    # Store the calculated ‘B‘ and extrapolated values in the convergence data
    convergence_data[f"{method}_b"] = b_values
    convergence_data[f"{method}_extrap"] = extrapolated_values



# Prepare data for FDM and FEM tables
rows = data_dict["convergence"]["num_sections"]
fdm_table_data = []
fem_table_data = []

for i, num_sections in enumerate(rows):
    # Append data for FDM
    fdm_table_data.append([
        analy_heat,
        convergence_data["fdm_2_values"][i],
        convergence_data["fdm_2_extrap"][i],
        convergence_data["fdm_2_errors"][i],
        convergence_data["fdm_2_b"][i],
        calc_error(convergence_data["fdm_2_extrap"][i],
            convergence_data["fdm_2_values"][i]) if i >= 2 else float('NaN'),
        calc_beta(convergence_data["fdm_2_extrap"][i], convergence_data["fdm_2_values"][i],
            convergence_data["fdm_2_values"][i-1], num_sections, rows[i-1]) if i >= 3 else
            float('NaN'),
```

```
        convergence_data["fdm_4_values"][i],
        convergence_data["fdm_4_extrap"][i],
        convergence_data["fdm_4_errors"][i],
        convergence_data["fdm_4_b"][i],
        calc_error(convergence_data["fdm_4_extrap"][i],
            convergence_data["fdm_4_values"][i]) if i >= 2 else float('NaN'),
        calc_beta(convergence_data["fdm_4_extrap"][i], convergence_data["fdm_4_values"][i],
            convergence_data["fdm_4_values"][i-1], num_sections, rows[i-1]) if i >= 3 else
            float('NaN'),
    ])

    # Append data for FEM
    fem_table_data.append([
        analy_heat,
        convergence_data["fem_2_values"][i],
        convergence_data["fem_2_extrap"][i],
        convergence_data["fem_2_errors"][i],
        convergence_data["fem_2_b"][i],
        calc_error(convergence_data["fem_2_extrap"][i],
            convergence_data["fem_2_values"][i]) if i >= 2 else float('NaN'),
        calc_beta(convergence_data["fem_2_extrap"][i], convergence_data["fem_2_values"][i],
            convergence_data["fem_2_values"][i-1], num_sections, rows[i-1]) if i >= 3 else
            float('NaN'),

        convergence_data["fem_4_values"][i],
        convergence_data["fem_4_extrap"][i],
        convergence_data["fem_4_errors"][i],
        convergence_data["fem_4_b"][i],
        calc_error(convergence_data["fem_4_extrap"][i],
            convergence_data["fem_4_values"][i]) if i >= 2 else float('NaN'),
        calc_beta(convergence_data["fem_4_extrap"][i], convergence_data["fem_4_values"][i],
            convergence_data["fem_4_values"][i-1], num_sections, rows[i-1]) if i >= 3 else
            float('NaN'),
    ])

    # Store extrapolated % errors for plotting
    if i >= 2:
        if "fdm_2_extrap_errors" not in convergence_data:
            convergence_data["fdm_2_extrap_errors"] = [float('NaN')] * 2
            convergence_data["fdm_4_extrap_errors"] = [float('NaN')] * 2
            convergence_data["fem_2_extrap_errors"] = [float('NaN')] * 2
            convergence_data["fem_4_extrap_errors"] = [float('NaN')] * 2
        convergence_data["fdm_2_extrap_errors"].append(calc_error(convergence_data["fdm_2_extrap"]
            convergence_data["fdm_2_values"][i]))
        convergence_data["fdm_4_extrap_errors"].append(calc_error(convergence_data["fdm_4_extrap"]
            convergence_data["fdm_4_values"][i]))
        convergence_data["fem_2_extrap_errors"].append(calc_error(convergence_data["fem_2_extrap"]
            convergence_data["fem_2_values"][i]))
        convergence_data["fem_4_extrap_errors"].append(calc_error(convergence_data["fem_4_extrap"]
            convergence_data["fem_4_values"][i]))
```

```
    else:
        if "fdm_2_extrap_errors" in convergence_data:
            convergence_data["fdm_2_extrap_errors"].append(float('NaN'))
            convergence_data["fdm_4_extrap_errors"].append(float('NaN'))
            convergence_data["fem_2_extrap_errors"].append(float('NaN'))
            convergence_data["fem_4_extrap_errors"].append(float('NaN'))


# Update columns for new fields
fdm_columns = [
    'True Q', 'FDM 2 Q', 'FDM 2 Extrap Q', 'FDM 2 % Error', 'FDM 2 Exact B',
    'FDM 2 Extrap % Error', 'FDM 2 Extrap B',
    'FDM 4 Q', 'FDM 4 Extrap Q', 'FDM 4 % Error', 'FDM 4 Exact B',
    'FDM 4 Extrap % Error', 'FDM 4 Extrap B'
]

fem_columns = [
    'True Q', 'FEM 2 Q', 'FEM 2 Extrap Q', 'FEM 2 % Error', 'FEM 2 Exact B',
    'FEM 2 Extrap % Error', 'FEM 2 Extrap B',
    'FEM 4 Q', 'FEM 4 Extrap Q', 'FEM 4 % Error', 'FEM 4 Exact B',
    'FEM 4 Extrap % Error', 'FEM 4 Extrap B'
]

fdm_df = pd.DataFrame(fdm_table_data, index=[f'N = {n}' for n in rows],
    columns=fdm_columns)
fem_df = pd.DataFrame(fem_table_data, index=[f'N = {n}' for n in rows],
    columns=fem_columns)

print(f"Convergence of FDM Heat with x_bar = {x_bar:.3f} and k_ratio = {k_ratio}")
print(fdm_df.to_string())
print("\n")

print(f"Convergence of FEM Heat with x_bar = {x_bar:.3f} and k_ratio = {k_ratio}")
print(fem_df.to_string())
print("\n")

# Plot convergence for FDM and FEM errors and extrapolated errors
ax_heat = axs_heat[idx]
num_sections = data_dict["convergence"]["num_sections"]

# Plot analytical error convergence for FDM and FEM, orders 2 and 4
ax_heat.plot(num_sections, convergence_data["fdm_2_errors"], 'o-', label="FDM 2 % Error
    (Analytical)")
ax_heat.plot(num_sections, convergence_data["fdm_4_errors"], 's-', label="FDM 4 % Error
    (Analytical)")
ax_heat.plot(num_sections, convergence_data["fem_2_errors"], '^-', label="FEM 2 % Error
    (Analytical)")
ax_heat.plot(num_sections, convergence_data["fem_4_errors"], 'd-', label="FEM 4 % Error
    (Analytical)")
```

```
            # Plot extrapolated error convergence for FDM and FEM, orders 2 and 4
            ax_heat.plot(num_sections, convergence_data["fdm_2_extrap_errors"], 'o--', label="FDM 2 %
                Error (Extrapolated)")
            ax_heat.plot(num_sections, convergence_data["fdm_4_extrap_errors"], 's--', label="FDM 4 %
                Error (Extrapolated)")
            ax_heat.plot(num_sections, convergence_data["fem_2_extrap_errors"], '^--', label="FEM 2 %
                Error (Extrapolated)")
            ax_heat.plot(num_sections, convergence_data["fem_4_extrap_errors"], 'd--', label="FEM 4 %
                Error (Extrapolated)")

            # Set log scales, labels, and legends
            ax_heat.set_xscale("log")
            ax_heat.set_yscale("log")
            ax_heat.set_xlabel("Number of Sections (N)")
            ax_heat.set_ylabel("% Error")
            ax_heat.set_title(f"Interface Heat Convergence (k_ratio={k_ratio})")
            ax_heat.legend()
            ax_heat.grid(True)


        fig_heat.tight_layout(rect=[0, 0.03, 1, 0.95])
        x_bar_path = "x_bar_1/" if x_bar_idx == 0 else "x_bar_2/"
        fig_heat.savefig(f"{file_path+x_bar_path}heat_convergence.png", dpi=300)




def clear_directory():
    # Make directories for plots
    import os
    import shutil

    base_dir = "images"
    sub_dirs = ["analytical", "convergence", "FDM", "FEM"]
    nested_dirs = ["x_bar_1", "x_bar_2"]

    # Create the base directory if it doesn't exist
    if not os.path.exists(base_dir):
        os.mkdir(base_dir)

    # Create or clear subdirectories and their nested directories
    for sub_dir in sub_dirs:
        section_path = os.path.join(base_dir, sub_dir)
        try:
            if os.path.exists(section_path):
                shutil.rmtree(section_path) # Remove all contents of the directory
            os.makedirs(section_path) # Recreate the directory
        except PermissionError as e:
            pass
```

```
        except Exception as e:
            pass

        # Create nested directories within each section
        for nested_dir in nested_dirs:
            nested_path = os.path.join(section_path, nested_dir)
            try:
                os.makedirs(nested_path) # Create the nested directory
            except PermissionError as e:
                pass
            except Exception as e:
                pass

def redirect_stdout():
    # Redirect all print statements to a file
    sys.stdout = open("output.txt", "w", encoding="utf-8")

def restore_stdout():
    # Restore original stdout and close the file
    sys.stdout.close()
    sys.stdout = sys.__stdout__


def main():
    # Delete previous images from other runs
    clear_directory()


    redirect_stdout() # Redirects the stdout to output.txt

    analytical() # Analytical solutions
    section_2(method="FDM") # FDM temperature and heat convection for different k ratios
    section_2(method="FEM") # FEM temperature and heat convection for different k ratios
    temp_convergence() # Convergence of FDM and FEM temperature
    heat_convergence() # Convergence of FDM and FEM temperature

    restore_stdout()



if __name__ == "__main__":
    main()
```

6.2 case1.py

```
import numpy as np
import matplotlib.pyplot as plt
```

```
from Bar import Bar, set_k_ratio

def get_analytical_constants(bar1:'Bar', bar2:'Bar', x_bar=2/np.pi, l=1):
    '''Solve for C1, D1, C2, and D2
    Returns np.array([C1, D1, C2, D2])'''
    epsilon = (bar1.k*bar1.area*bar1.alpha) / (bar2.k*bar2.area*bar2.alpha)
    A = np.array([
        #C1                                  D1            C2                              D2
        [0,                                  1,            0,                              0],
        [0,                                  0,            np.sinh(bar2.alpha*l),
            np.cosh(bar2.alpha*l)],
        [np.sinh(bar1.alpha*x_bar),           0,           -1*np.sinh(bar2.alpha*x_bar),
            -1*np.cosh(bar2.alpha*x_bar)],
        [epsilon*np.cosh(bar1.alpha*x_bar),  0,            -1*np.cosh(bar2.alpha*x_bar),
            -1*np.sinh(bar2.alpha*x_bar)]
    ])

    B = np.array([
        0,
        100,
        0,
        0
    ])

    return np.linalg.solve(A, B)

def get_analytical_datapoints(x_points, bar1:'Bar', bar2:'Bar', x_bar=2/np.pi, l=1):
    '''Generates the temperature distribution given x_points.
    Returns np.array of tempature values'''
    constants = get_analytical_constants(bar1=bar1,bar2=bar2,x_bar=x_bar,l=1)
    C1, D1, C2, D2 = constants[0], constants[1], constants[2], constants[3]

    # Two temp functions for the left and right side of the interface
    left_temp = lambda x: C1*np.sinh(bar1.alpha*x) + D1*np.cosh(bar1.alpha*x)
    right_temp = lambda x: C2*np.sinh(bar2.alpha*x) + D2*np.cosh(bar2.alpha*x)

    temp_values = np.array([])
    for x in x_points:
        if x <= x_bar:
            temp = left_temp(x)
        else:
            temp = right_temp(x)
        temp_values = np.append(temp_values, temp)

    return temp_values

def get_analytical_heat_convection(x, bar1:'Bar', bar2:'Bar', x_bar=2/np.pi, l=1):
    '''Gets the analytical heat convection at x from heat flux conservation'''
    constants = get_analytical_constants(bar1=bar1,bar2=bar2,x_bar=x_bar,l=1)
```

```
    C1, D1, C2, D2 = constants[0], constants[1], constants[2], constants[3]

    t_prime = C2*bar2.alpha*np.cosh(bar2.alpha*x) + D2*bar2.alpha*np.sinh(bar2.alpha*x)

    q_dot = -bar2.k*bar2.area*t_prime
    return q_dot

def fdm_array(bar1:'Bar', bar2:'Bar', num_sections: int=6, x_bar=2/np.pi, l=1, order=2):
    '''num_sections = total sections to cut the bar. Note that this is NOT evenly distributed over
        the entire
    bar. Half of the sections are left of x_bar, half are to the right.
    Returns the A matrix of the FDM equation Ax = B'''
    dx1 = x_bar / (num_sections / 2)
    dx2 = (l - x_bar) / (num_sections / 2)

    if order == 2:
        w1 = dx1
        w2 = dx2

        epsilon1 = 1 + dx1**2 * bar1.alpha**2 / 2
        epsilon2 = 1 + dx2**2 * bar2.alpha**2 / 2

        beta1 = bar1.k * bar1.area / w1
        beta2 = bar2.k * bar2.area / w2

        k1 = 2*beta1*epsilon1
        k2 = 2*beta2*epsilon2

        rho = beta1*epsilon1 + beta2*epsilon2
    elif order == 4:
        w1 = dx1 + dx1**3 * bar1.alpha**2 / 6
        w2 = dx2 + dx2**3 * bar2.alpha**2 / 6

        epsilon1 = 1 + dx1**2 * bar1.alpha**2 / 2 + dx1**4 * bar1.alpha**4 / 24
        epsilon2 = 1 + dx2**2 * bar2.alpha**2 / 2 + dx2**4 * bar2.alpha**4 / 24

        beta1 = bar1.k * bar1.area / w1
        beta2 = bar2.k * bar2.area / w2

        k1 = 2*beta1*epsilon1
        k2 = 2*beta2*epsilon2

        rho = beta1*epsilon1 + beta2*epsilon2

    # Making the matrix
    # There are num_sections - 1 rows and columns
    # row 0, row n - 2, and row num_sections/2 - 1 are unique
    # the rest are -1, k, -1 where k changes from k1 to k2 at num_sections/2

    # Initialize matrix A with zeros
```

```
    A = np.zeros((num_sections - 1, num_sections - 1))

    # Fill the matrix A
    for row in range(num_sections - 1):
        if row == 0:
            # First row for boundary at x = 0 (for example Dirichlet or Neumann condition)
            A[row, row] = k1
            A[row, row+1] = -beta1
        elif row == num_sections // 2 - 1:
            # Special row at the interface between bar1 and bar2
            A[row, row-1] = -beta1
            A[row, row] = rho
            A[row, row+1] = -beta2
        elif row == num_sections - 2:
            # Last row for boundary at x = l (again assuming Dirichlet or Neumann condition)
            A[row, row-1] = -beta2
            A[row, row] = k2
        else:
            # Interior rows for bar1 and bar2 (uniform grid)
            if row < num_sections // 2 - 1:
                # In bar1 region
                A[row, row-1] = -beta1
                A[row, row] = k1
                A[row, row+1] = -beta1
            else:
                # In bar2 region
                A[row, row-1] = -beta2
                A[row, row] = k2
                A[row, row+1] = -beta2
    return A

def fem_array(bar1:'Bar', bar2:'Bar', num_sections: int=6, x_bar=2/np.pi, l=1):
    '''num_sections = total sections to cut the bar. Note that this is NOT evenly distributed over
        the entire
    bar. Half of the sections are left of x_bar, half are to the right.
    Returns the A matrix of the FEM equation Ax = B'''
    dx1 = x_bar / (num_sections / 2)
    dx2 = (l - x_bar) / (num_sections / 2)

    k1 = (2/dx1 + 4*bar1.alpha**2*dx1/6) / (1/dx1 - bar1.alpha**2*dx1/6)
    k2 = (2/dx2 + 4*bar2.alpha**2*dx2/6) / (1/dx2 - bar2.alpha**2*dx2/6)
    beta1 = (bar1.k * bar1.area) / (dx1)
    beta2 = (bar2.k * bar2.area) / (dx2)
    rho = beta1 + beta2 + (bar1.k * bar1.area * dx1 * bar1.alpha**2) / 2 + (bar2.k * bar2.area * dx2
        * bar2.alpha**2) / 2

    # Making the matrix
    # There are num_sections - 1 rows and columns
    # row 0, row n - 2, and row num_sections/2 - 1 are unique
    # the rest are -1, k, -1 where k changes from k1 to k2 at num_sections/2
```

```
    # Initialize matrix A with zeros
    A = np.zeros((num_sections - 1, num_sections - 1))

    # Fill the matrix A
    for row in range(num_sections - 1):
        if row == 0:
            # First row for boundary at x = 0 (for example Dirichlet or Neumann condition)
            A[row, row] = k1
            A[row, row+1] = -1
        elif row == num_sections // 2 - 1:
            # Special row at the interface between bar1 and bar2
            A[row, row-1] = -beta1
            A[row, row] = rho # This is the rho value you computed
            A[row, row+1] = -beta2
        elif row == num_sections - 2:
            # Last row for boundary at x = l (again assuming Dirichlet or Neumann condition)
            A[row, row-1] = -1
            A[row, row] = k2
        else:
            # Interior rows for bar1 and bar2 (uniform grid)
            if row < num_sections // 2 - 1:
                # In bar1 region
                A[row, row-1] = -1
                A[row, row] = k1
                A[row, row+1] = -1
            else:
                # In bar2 region
                A[row, row-1] = -1
                A[row, row] = k2
                A[row, row+1] = -1
    return A

def get_fem_p2_k(row, col, bar:'Bar', dx):
    if (row == 1 and col == 1) or (row == 2 and col == 2):
        return bar.k * bar.area / dx + bar.h * bar.p * dx / 3.0
    elif row == 3 and col == 3:
        return 1 * bar.k * bar.area / (3*dx) + bar.h * bar.p * dx / 30
    elif (row == 1 and col == 2) or (row == 2 and col == 1):
        return -1 * bar.k * bar.area / dx + bar.h * bar.p * dx / 6
    elif (row == 1 and col == 3) or (row == 3 and col == 1) or (row == 2 and col == 3) or (row == 3
        and col == 2):
        return bar.h * bar.p * dx / 12

def get_all_fem_p2_k(bar:'Bar', dx):
    K = np.zeros((3, 3))
    for row in range(3):
        for col in range(3):
            K[row, col] = get_fem_p2_k(row+1, col+1, bar, dx)
    return K
```

```
def fem_p_2_array(bar1:'Bar', bar2:'Bar', num_sections: int=6, x_bar=2/np.pi, l=1):
    '''num_sections = total sections to cut the bar. Note that this is NOT evenly distributed over
        the entire
    bar. Half of the sections are left of x_bar, half are to the right.
    Returns the A matrix of the FEM equation Ax = B'''
    dx1 = x_bar / (num_sections / 2)
    dx2 = (l - x_bar) / (num_sections / 2)

    K_1 = get_all_fem_p2_k(bar1, dx1)
    K_2 = get_all_fem_p2_k(bar2, dx2)

    # Initialize matrix A with zeros
    rows = 2 * num_sections - 1
    cols = 2 * num_sections - 1
    A = np.zeros((rows, cols))

    omega_col_index = num_sections - 1
    num_omega = num_sections
    for i in range(num_sections):
        row1 = i*2
        row2 = row1 + 1
        if i == 0:
            A[row1][0] = K_1[2-1, 2-1] + K_1[1-1, 1-1]
            A[row1][1] = K_1[1-1, 2-1]
            A[row1][omega_col_index] = K_1[2-1, 3-1]
            A[row1][omega_col_index+1] = K_1[1-1, 3-1]

            A[row2][0] = K_1[3-1, 2-1]
            A[row2][omega_col_index] = K_1[3-1, 3-1]
        elif i == num_sections - 2: # Second last pair of rows
            A[row1][i-1] =               K_2[2-1, 1-1]
            A[row1][i] = K_2[2-1, 2-1] + K_2[1-1, 1-1]
            A[row1][omega_col_index+i] = K_2[2-1, 3-1]
            A[row1][omega_col_index+i+1] = K_2[1-1, 3-1]

            A[row2][i-1] = K_2[3-1, 1-1]
            A[row2][i] = K_2[3-1, 2-1]
            A[row2][omega_col_index+i] = K_2[3-1, 3-1]
        elif i == num_sections - 1: # Last pair of rows, use only omega equation
            A[row1][omega_col_index-1] = K_2[3-1, 1-1]
            A[row1][-1] = K_2[3-1, 3-1]
        elif i == num_sections // 2 - 1: # Interface element
            A[row1][i-1] = K_1[2-1, 1-1]
            A[row1][i] = K_1[2-1, 2-1] + K_2[1-1, 1-1]
            A[row1][i+1] = K_2[1-1, 2-1]
            A[row1][omega_col_index + i] = K_1[2-1, 3-1]
            A[row1][omega_col_index + i + 1] = K_2[1-1, 3-1]

            A[row2][i-1] = K_1[3-1, 1-1]
```

```
            A[row2][i] = K_1[3-1, 2-1]
            A[row2][omega_col_index + i] = K_1[3-1, 3-1]
        else:
            if i < num_sections // 2: # We are in the first bar
                A[row1][i-1] = K_1[2-1, 1-1]
                A[row1][i] = K_1[2-1, 2-1] + K_1[1-1, 1-1]
                A[row1][i+1] = K_1[1-1, 2-1]
                A[row1][omega_col_index + i] = K_1[2-1, 3-1]
                A[row1][omega_col_index + i + 1] = K_1[1-1, 3-1]

                A[row2][i-1] = K_1[3-1, 1-1]
                A[row2][i] = K_1[3-1, 2-1]
                A[row2][omega_col_index + i] = K_1[3-1, 3-1]
            else:  # We are in the second bar
                A[row1][i-1] = K_2[2-1, 1-1]
                A[row1][i] = K_2[2-1, 2-1] + K_2[1-1, 1-1]
                A[row1][i+1] = K_2[1-1, 2-1]
                A[row1][omega_col_index + i] = K_2[2-1, 3-1]
                A[row1][omega_col_index + i + 1] = K_2[1-1, 3-1]

                A[row2][i-1] = K_2[3-1, 1-1]
                A[row2][i] = K_2[3-1, 2-1]
                A[row2][omega_col_index + i] = K_2[3-1, 3-1]
    return A



def solve(bar1:'Bar', bar2:'Bar', num_sections: int=6, x_bar=2/np.pi, l=1, method="FDM", order=2):
    '''Solves using FDM or FEM as specified. Returns (temps, x_values)'''
    if method == "FDM":
        A = fdm_array(bar1,
                      bar2,
                      num_sections=num_sections,
                      x_bar=x_bar,
                      l=l,
                      order=order)
        # Generate the B vector
        if order == 2:
            dx2 = (l - x_bar) / (num_sections / 2)
            w2 = dx2
            beta2 = bar2.k * bar2.area / w2
        elif order == 4:
            dx2 = (l - x_bar) / (num_sections / 2)
            w2 = dx2 + dx2**3 * bar2.alpha**2 / 6
            beta2 = bar2.k * bar2.area / w2

        B = np.zeros((num_sections - 1))
        B[-1] = beta2*100

        # Add boundary conditions
```

```
    interior_temps = np.linalg.solve(A, B)
    all_temps = np.array([0])
    all_temps = np.append(all_temps, interior_temps)
    all_temps = np.append(all_temps, 100)

    # Generate the x_values to return along with the temps
    dx2 = (l - x_bar) / (num_sections / 2)
    x_values = np.linspace(0, x_bar, num_sections // 2 + 1) # [0, x_bar]
    x_values = np.append(x_values, np.linspace(x_bar+dx2, l, num_sections // 2))

elif method == "FEM":
    if order == 2:
        A = fem_array(bar1,
                  bar2,
                  num_sections=num_sections,
                  x_bar=x_bar,
                  l=l)
        # Generate the B vector
        B = np.zeros((num_sections - 1))
        B[-1] = 100

        # Add boundary conditions
        interior_temps = np.linalg.solve(A, B)
        all_temps = np.array([0])
        all_temps = np.append(all_temps, interior_temps)
        all_temps = np.append(all_temps, 100)

        # Generate the x_values to return along with the temps
        dx2 = (l - x_bar) / (num_sections / 2)
        x_values = np.linspace(0, x_bar, num_sections // 2 + 1) # [0, x_bar]
        x_values = np.append(x_values, np.linspace(x_bar+dx2, l, num_sections // 2))
    elif order == 4:
        A = fem_p_2_array(bar1,
                  bar2,
                  num_sections=num_sections,
                  x_bar=x_bar,
                  l=l)

        # Generate the B vector
        dx2 = (l - x_bar) / (num_sections / 2)
        B = np.zeros((num_sections*2-1))
        K_2 = get_all_fem_p2_k(bar2, dx2)
        B[0] = 0
        B[1] = 0
        B[-3] = -1 * K_2[1-1, 2-1] * 100
        B[-2] = 0
        B[-1] = -1 * K_2[3-1, 2-1] * 100

        # Add boundary conditions
        solution = np.linalg.solve(A, B)
```

```
            all_temps = np.array([0])
            all_temps = np.append(all_temps, solution[0:num_sections-1])
            all_temps = np.append(all_temps, 100)

            # Generate the x_values to return along with the temps
            dx2 = (l - x_bar) / (num_sections / 2)
            x_values = np.linspace(0, x_bar, num_sections // 2 + 1) # [0, x_bar]
            x_values = np.append(x_values, np.linspace(x_bar+dx2, l, num_sections // 2))

    return all_temps, x_values


if __name__ == "__main__":
    bar1 = Bar(
        radius=0.1,
        k=0.5,
        h=0.25
    )
    bar2 = Bar(
        radius=0.1,
        k=0.5,
        h=0.25
    )

    #set_k_ratio(8, bar1, bar2)


    x_bar = 0.7

    x_values = np.linspace(0, 1, 1000)
    temp_values = get_analytical_datapoints(x_points=x_values,
                                            bar1=bar1,
                                            bar2=bar2,
                                            x_bar=x_bar)


    # fdm tests
    l = 1
    num_sections = 16
    dx1 = x_bar / (num_sections / 2)
    dx2 = (l - x_bar) / (num_sections / 2)
    fdm_temps, x_fdm = solve(bar1=bar1,
                   bar2=bar2,
                   num_sections=num_sections,
                   x_bar=x_bar,
                   l=1,
                   method="FEM",
                   order=2)
    fem_temps, x_fem = solve(bar1=bar1,
                   bar2=bar2,
```

```
                num_sections=num_sections,
                x_bar=x_bar,
                l=1,
                method="FEM",
                order=4)

    print(f"A: {bar1.area}")
    print(f"k: {bar1.k}")
    print(f"h: {bar1.h}")
    print(f"P: {bar1.p}")
    print(f"dx: {dx1}")


    # Plot the data
    plt.plot(x_values, temp_values, label='Temperature Distribution')
    plt.axvline(x=x_bar, color='r', linestyle='--', label='x_bar')
    plt.plot(x_fdm, fdm_temps, 'o-', label='Temperature Distribution (FDM)', color='g')
    plt.plot(x_fem, fem_temps, 'o-', label='Temperature Distribution (FEM)', color='r')
    plt.xlabel('x')
    plt.ylabel('Temperature')
    plt.title('Temperature Distribution Along the Bar')
    plt.legend()
    plt.grid(True)

    # Show plot
    plt.show()
```

6.3 Bar.py

```
# Bar.py

from numpy import pi

class Bar():
    def __init__(self, radius:float=0.1, k:float=0.5, h:float=0.0025):
        self.radius:float = radius
        self.k:float = k
        self.h:float = h

        self.update_properties()

    def update_properties(self):
        self.area:float = pi*self.radius**2
        self.p: float = 2*pi*self.radius
        self.alpha = ((self.h*self.p)/(self.k*self.area))**0.5

    def set_k(self, k:float):
        self.k = k
```

```
        self.update_properties()

def set_k_ratio(ratio: float, bar1:'Bar', bar2:'Bar'):
    '''Sets the value of bar2.k = ratio*bar1.k and updates internal properties'''
    bar2.set_k(bar1.k*ratio)
```

6.4 common.py

```
import numpy as np
from Bar import Bar

def fdm_heat_extraction(t_0, t_1, dx, bar:'Bar', order=2):
    '''Get the heat conduction at the point t_1 using Taylor series'''
    if order == 1:
        return -1 * bar.k * bar.area * (t_1 - t_0) / dx
    elif order == 2:
        return -1 * bar.k * bar.area * (((t_1 - t_0) / dx) + (bar.alpha**2 * dx * t_1 / 2))
    elif order == 4:
        return -1 * bar.k * bar.area * ((144*t_1 - 144*t_0 + 72*dx**2*bar.alpha**2*t_1 +
            6*dx**4*bar.alpha**4*t_1) / (144 * dx + 24*dx**3 * bar.alpha**2))

def fem_heat_extraction(t_0, t_1, dx, bar:'Bar', order=2):
    '''Get the heat conduction at the point t_1 using FEM equation'''
    if order == 2:
        # term_1 = (-1/dx + bar.alpha**2*dx/6) * t_0
        # term_2 = (1/dx + 2*bar.alpha**2*dx/6) * t_1
        term_1 = (t_1 - t_0) / dx
        term_2 = bar.alpha**2 * dx * t_1 / 2
        return -1 * bar.k * bar.area * (term_1 + term_2)
    elif order == 4:
        term_1 = (-1/dx + bar.alpha**2*dx/6) * t_0
        term_2 = (1/dx + 2*bar.alpha**2*dx/6) * t_1
        return -1 * bar.k * bar.area * (term_1 + term_2)

def heat_extraction(t_0, t_1, dx, bar:'Bar', order=2, method="FDM"):
    if method == "FDM":
        return fdm_heat_extraction(t_0, t_1, dx, bar, order)
    elif method == "FEM":
        return fdm_heat_extraction(t_0, t_1, dx, bar, order)

def calc_error(exact, q_1):
    return np.abs((exact - q_1) / exact)

def calc_beta(exact, q_1, q_2, dx_1, dx_2):
    return np.log(np.abs((exact - q_1)/(exact - q_2))) / np.log(dx_1 / dx_2)

def calc_extrapolated(q1, q2, q3, tolerance=1e-10):
```

```
'''Calculate Richardson extrapolation, returns NaN if denominator is too small.'''
numerator = q1 * q3 - q2**2
denominator = q1 + q3 - 2 * q2
if abs(denominator) < tolerance:
    return float('NaN') # Return NaN if denominator is close to zero
return numerator / denominator
```