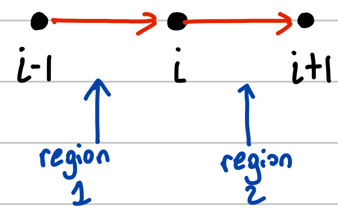
Judson Upchurch

4th Order FDM

Constitutive EQ: $k_1 A_1 T_i^{\prime -} - k_2 A_2 T_i^{\prime +} = 0$



$$T'' - \alpha^2 T = 0 \quad \therefore \quad T'' = \alpha^2 T$$
$$T''' = \alpha^2 T'$$
$$T'''' = \alpha^2 T'' = \alpha^4 T$$

$$T_{i+1} = T_i + \Delta x T_i' + \frac{\Delta x^2}{2} T_i'' + \frac{\Delta x^3}{6} T_i''' + \frac{\Delta x^4}{24} T_i''''$$

$$T_{i+1} = T_i + \Delta x T_i' + \frac{\Delta x^2}{2} \alpha^2 T_i + \frac{\Delta x^3}{6} \alpha^2 T_i' + \frac{\Delta x^4}{24} \alpha^4 T$$

$$\Delta x T_i' + \frac{\Delta x^3}{6} \alpha^2 T_i' = T_{i+1} - T_i - \frac{\Delta x^2 \alpha^2}{2} T_i - \frac{\Delta x^4}{24} \alpha^4 T_i$$

$$\underbrace{\left[\Delta x_2 + \frac{\Delta x_2^3 \alpha_2^2}{6}\right]}_{w_2} T_i' = T_{i+1} - \underbrace{\left[+1 + \frac{\Delta x_2^2 \alpha_2^2}{2} + \frac{\Delta x_2^4}{24} \alpha_2^4\right]}_{\varepsilon_2} T_i$$

$$\boxed{T_i^{\prime +} = \frac{1}{w_2} T_{i+1} - \frac{\varepsilon_2}{w_2} T_i} \quad T' \text{ from right}$$

$$T_{i-1} = T_i - \Delta x T_i' + \frac{\Delta x^2}{2} \alpha^2 T_i - \frac{\Delta x^3 \alpha^2}{6} T_i' + \frac{\Delta x^4 \alpha^4}{24} T_i$$

$$\Delta x T_i' + \frac{\Delta x^3 \alpha^2}{6} T_i' = -T_{i-1} + T_i + \frac{\Delta x^2}{2} \alpha^2 T_i + \frac{\Delta x^4 \alpha^4}{24} T_i$$

$$\underbrace{\left[\Delta x_1 + \frac{\Delta x_1^3 \alpha_1^2}{6}\right]}_{w_1} T_i' = -T_{i-1} + \underbrace{\left[1 + \frac{\Delta x_1^2 \alpha_1^2}{2} + \frac{\Delta x_1^4 \alpha_1^4}{24}\right]}_{\varepsilon_1} T_i$$

$$\boxed{T_i^{\prime -} = \frac{-1}{w_1} T_{i-1} + \frac{\varepsilon_1}{w_1} T_i} \quad T' \text{ from left}$$

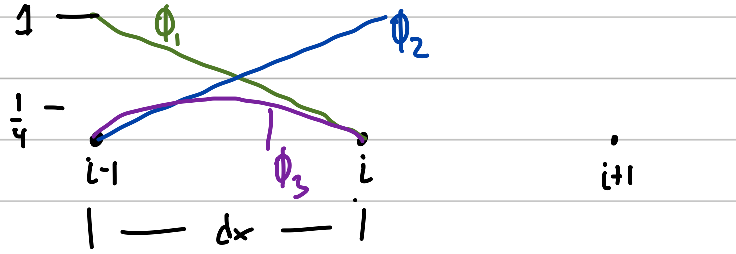
with $k_1 A_1 T_i'^{-} - k_2 A_2 T_i'^{+} = 0$

$$k_1 A_1 \left[\frac{-1}{w_1} T_{i-1} + \frac{\varepsilon_1}{w_1} T_i \right] - k_2 A_2 \left[\frac{1}{w_2} T_{i+1} - \frac{\varepsilon_2}{w_2} T_i \right] = 0$$

$$\underbrace{-\frac{k_1 A_1}{w_1}}_{\beta_1} T_{i-1} + \left[\frac{k_1 A_1 \varepsilon_1}{w_1} + \frac{k_2 A_2 \varepsilon_2}{w_2} \right] T_i \underbrace{-\frac{k_2 A_2}{w_2}}_{\beta_2} T_{i+1} = 0$$

$$\begin{bmatrix} \beta_1 \varepsilon_1 + \beta_2 \varepsilon_2 & -\beta_2 & 0 \\ -\beta_1 & \beta_1 \varepsilon_1 + \beta_2 \varepsilon_2 & -\beta_2 \\ & & \\ & & \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ \\ \end{bmatrix} = \begin{bmatrix} \beta_1 T_1 \\ 0 \\ 0 \\ \\ \end{bmatrix}$$

FEM



$$\int_I \dot{Q}' + hPT\, dx = 0$$

$$\int_I \dot{Q}' v\, dx + \int_I hPTv\, dx = 0$$

$$\phi_1 = 1 - \frac{x}{\Delta x} \qquad \phi_2 = \frac{x}{\Delta x} \qquad \phi_3 = \phi_1 \cdot \phi_2 = \frac{x}{\Delta x} - \frac{x^2}{\Delta x^2}$$

$$\phi_1' = -\frac{1}{\Delta x} \qquad \phi_2' = \frac{1}{\Delta x}$$

$$\phi_3' = \frac{1}{\Delta x} - \frac{2x}{\Delta x^2}$$

$$\int_I (\dot{Q}v)'\, dx - \int_I \dot{Q}v'\, dx + hP\int_I Tv\, dx = 0$$

some unknown

$$T(x) = T_{i-1}\phi_1 + T_i\phi_2 + \Omega\phi_3$$

$$\dot{Q}v\Big|_I - \int_I \dot{Q}v'\, dx + hP\int_I Tv\, dx = 0$$

$$T'(x) = T_{i-1}\phi_1' + T_i\phi_2' + \Omega\phi_3'$$

$$-\int_I \dot{Q}\phi_m'\, dx + hP\int_I T\phi_m\, dx = -\dot{Q}\phi_m\Big|_I$$

$$\dot{Q} = -kAT'$$

$$-k_iA_i\int_I \left(T_{i-1}\phi_1'\phi_m' + T_i\phi_2'\phi_m' + \Omega\phi_3'\phi_m'\right) dx + hP\int_I \left(T_{i-1}\phi_1\phi_m + T_i\phi_2\phi_m + \Omega\phi_3\phi_m\right) dx = -\dot{Q}\phi_m\Big|_I$$

for $m = 1, 2, 3$

$$m=1: -k_iA_i\left[T_{i-1}\int\phi_1'\phi_1' + T_i\int\phi_2'\phi_1' + \Omega\int\phi_3'\phi_1'\right] + hP_i\left[T_{i-1}\int\phi_1\phi_1 + T_i\int\phi_2\phi_1 + \Omega\int\phi_3\phi_1\right] = -\left[\dot{Q}^+_{i-1}\right]$$

$$m=2: -k_iA_i\left[T_{i-1}\int\phi_1'\phi_2' + T_i\int\phi_2'\phi_2' + \Omega\int\phi_3'\phi_2'\right] + hP_i\left[T_{i-1}\int\phi_1\phi_2 + T_i\int\phi_2\phi_2 + \Omega\int\phi_3\phi_2\right] = -\left[-\dot{Q}^-_{i}\right]$$

$$m=3: -k_iA_i\left[T_{i-1}\int\phi_1'\phi_3' + T_i\int\phi_2'\phi_3' + \Omega\int\phi_3'\phi_3'\right] + hP_i\left[T_{i-1}\int\phi_1\phi_3 + T_i\int\phi_2\phi_3 + \Omega\int\phi_3\phi_3\right] = 0$$

$$m=1: \quad k_{11} T_{i-1} + k_{12} T_i + k_{13} \Omega = -\dot{Q}^{+}_{i-1}$$

$$m=2: \quad k_{21} T_{i-1} + k_{22} T_i + k_{23} \Omega = +\dot{Q}^{-}_{i-1}$$

$$m=3: \quad k_{31} T_{i-1} + k_{32} T_i + k_{33} \Omega = 0$$

$$k_{m,n} = k_1 A_1 \int \phi_m' \phi_n' \, dx + h P_1 \int \phi_m \phi_n \, dx$$

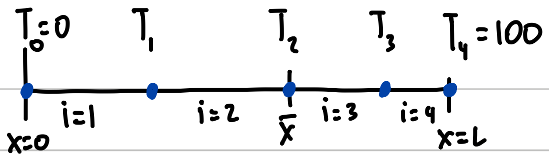
$$\begin{bmatrix} k^i_{11} & k^i_{12} & k^i_{13} \\ k^i_{21} & k^i_{22} & k^i_{23} \\ k^i_{31} & k^i_{32} & k^i_{33} \end{bmatrix} \begin{bmatrix} T_{i-1} \\ T_i \\ \Omega_i \end{bmatrix} = \begin{bmatrix} -\dot{Q}^{+}_{i-1} \\ \dot{Q}^{-}_i \\ 0 \end{bmatrix}$$

$$-\dot{Q}^{+}_i + \dot{Q}^{-}_i = 0$$

$$\begin{bmatrix} k^{i+1}_{11} & k^{i+1}_{12} & k^{i+1}_{13} \\ k^{i+1}_{21} & k^{i+1}_{22} & k^{i+1}_{23} \\ k^{i+1}_{31} & k^{i+1}_{32} & k^{i+1}_{33} \end{bmatrix} \begin{bmatrix} T_i \\ T_{i+1} \\ \Omega_{i+1} \end{bmatrix} = \begin{bmatrix} -\dot{Q}^{+}_i \\ \dot{Q}^{-}_{i+1} \\ 0 \end{bmatrix}$$

$$\boxed{k^i_{21} \cdot T_{i-1} + \left(k^i_{22} + k^{i+1}_{11}\right) T_i + \left(k^{i+1}_{12}\right) T_{i+1} + k^i_{23} \cdot \Omega_i + k^{i+1}_{13} \cdot \Omega_{i+1} = 0}$$

$$\boxed{k^i_{31} \cdot T_{i-1} + \left(k^i_{32} + k^{i+1}_{31}\right) T_i + \left(k^{i+1}_{32}\right) T_{i+1} + k^i_{33} \cdot \Omega_i + k^{i+1}_{33} \, \Omega_{i+1} = 0}$$



$$\boxed{k^i_{21} \cdot T_{i-1} + \left(k^i_{22} + k^{i+1}_{11}\right) T_i + \left(k^{i+1}_{12}\right) T_{i+1} + k^i_{23} \cdot \Omega_i + k^{i+1}_{13} \cdot \Omega_{i+1} = 0}$$

$$\boxed{k^i_{31} \cdot T_{i-1} + k^i_{32} \cdot T_i + k^i_{33} \Omega_i = 0}$$

$N=4$ sections, 5 nodes

$$\begin{array}{c} \\ i=1 \\ i=1 \\ i=2 \\ i=2 \\ i=3 \\ i=3 \\ i=4 \\ i=4 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} T_1 & T_2 & T_3 & \Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 \end{array} \\
\left[\begin{array}{c:c:c:c:c:c:c}
k^1_{22} + k^2_{11} & k^2_{12} & & k^1_{23} & k^2_{13} & & \\
k^1_{32} & & & k^1_{33} & & & \\
k^2_{21} & k^2_{22} + k^3_{11} & k^2_{12} & & k^2_{23} & k^3_{13} & \\
k^2_{31} & k^2_{32} & & & k^2_{33} & & \\
 & k^3_{21} & k^3_{22} + k^4_{11} & & & k^3_{23} & k^4_{13} \\
 & k^3_{31} & k^3_{32} & & & k^3_{33} & \\
 & & k^4_{21} & & & & \\
 & & k^4_{31} & & & & k^4_{33}
\end{array}\right]
\end{array}
\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{bmatrix}
=
\begin{bmatrix} -k^1_{21} \cdot T_0 \\ -k^1_{13} \cdot T_0 \\ 0 \\ 0 \\ -k^3_{12} \cdot T_4 \\ 0 \\ -\cancel{(k^4_{22} + k^5_{11})} \cdot T_4 \\ -k^4_{32} \cdot T_4 \end{bmatrix}$$

needs T_5

$$k_{m,n} = k_1 A_1 \int_0^{\Delta x} \phi'_m \phi'_n \, dx + hP_1 \int_0^{\Delta x} \phi_m \phi_n \, dx$$

$\phi_1 = 1 - \frac{x}{\Delta x}$ $\quad \phi_2 = \frac{x}{\Delta x}$ $\quad \phi_3 = \phi_1 \cdot \phi_2 = \frac{x}{\Delta x} - \frac{x^2}{\Delta x^2}$

$\phi'_1 = -\frac{1}{\Delta x}$ $\quad \phi'_2 = \frac{1}{\Delta x}$

$\phi'_3 = \frac{1}{\Delta x} - \frac{2x}{\Delta x^2}$

$$k^i_{11} = k_i A_i \int_0^{\Delta x} \frac{1}{\Delta x^2} dx + hP_i \int_0^{\Delta x} \left(1 - \frac{x}{\Delta x}\right)^2 dx$$

$$\boxed{k^i_{11} = \frac{k_i A_i}{\Delta x_i} + \frac{hP_i \Delta x_i}{3}}$$

$$\boxed{k^i_{12} = k^i_{21} = -\frac{k_i A_i}{\Delta x_i} + \frac{hP_i \Delta x_i}{6}}$$

$$k^i_{22} = k_i A_i \int_0^{\Delta x} \frac{1}{\Delta x} dx + hP_i \int_0^{\Delta x} \left(\frac{x}{\Delta x}\right)^2 dx$$

$$\boxed{k^i_{13} = k^i_{31} = 0 + \frac{hP_i \Delta x_i}{12}}$$

$$\boxed{k^i_{22} = \frac{k_i A_i}{\Delta x_i} + \frac{hP_i \Delta x_i}{3}}$$

$$\boxed{k^i_{33} = \frac{k_i A_i}{3 \Delta x_i} + \frac{hP_i \Delta x_i}{30}}$$

$$\boxed{k^i_{23} = k^i_{32} = 0 + \frac{hP_i \Delta x_i}{12}}$$