AERO 430 HW 3

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1 Analytical Solution

1.1 Abstract

This section provides an analytical solution for the temperature distribution of a bimaterial beam connected at an interface point. The beam is constrained such that the environment is at 0 degrees, the right end of the beam is fixed at 100 degrees, and the left is fixed at 0 degrees.
In this report, the beam consists of two sections with the same area but different thermal conductivities. The affects of varying the ratio between the thermal conductivities will be analyzed as well as changing the interface location.

1.2 Derivation of General Solution

Let us first look at derivation for the general solution for temperature as a function along the rod. To begin, the governing equation will be conservation of heat flux.

1.2.1 Conservation of Heat Flux

In the case of a rod without internal heat generation, conservation of heat flux reduces to the total heat flux entering the system is equal and opposite to the heat flux leaving the system.

$$\dot{Q}_{out}(x) - \dot{Q}_{in}(x) = 0 \tag{1}$$

$\dot{Q}$ is the rate of heat transfer in energy / second units. We will use Joules/second, aka Watts.

1.2.2 Newton's Law of Cooling

Newton's Law of Cooling shows that the rate of heat transfer to the surroundings is proportional to the area of heat transfer and the temperature difference between the object and the surroundings.

$$\dot{Q}_{env}(x) = hA\left(T(x) - T_{amb}\right) \tag{2}$$

$\dot{Q}_{env}(x)$ is the rate of heat flux to the environment at a point along the rod in Joules/second. h is the heat transfer coefficient of the environment's fluid in $\frac{W}{m^2\,^\circ C}$. A is the area exposed to the surroundings in m^2. $T(x)$ is the temperature of the rod at position x in $^\circ C$.

In both of our cases, the ambient temperature, T_{amb} is 0 $^\circ C$. This reduces the equation for Newton's Law of Cooling to:

$$\dot{Q}_{env}(x) = hAT(x) \tag{3}$$

1.2.3 Fourier's Law of Conduction

Fourier's Law of Conduction states that the rate of heat transfer through a material via conduction is proportional to the area of heat transfer and the negative temperature gradient.

$$\dot{Q}_c(x) = -kA\frac{dT(x)}{dx} \tag{4}$$

$\dot{Q}_c(x)$ is the rate of heat transfer via conduction through the material in Joules/second. k is the thermal conductivity of the rod in $\frac{W}{m^\circ C}$. A is the cross-sectional area of the rod in m^2. $\frac{dT(x)}{dx}$ is the temperature gradient of the rod in $\frac{^\circ C}{m}$.

1.2.4 General Solution

If we look at an infinitesimal slice of the rod starting at position x and total length dx, we use conservation of heat flux (Equation 1) to get:

$$\dot{Q}(x+dx) - \dot{Q}(x) + \dot{Q}_{lat}(x;dx) = 0$$

From left to right, we have the heat flux leaving the right side of the rod slice, the heat flux entering the left side of the rod slice, and the heat flux leaving the rod slice from the outer circumference to the environment. Replacing $\dot{Q}_{lat}(x;dx)$ with $hPdxT(x)$ from Newton's Law of Cooling (Equation 3) gives:

$$\dot{Q}(x+dx) - \dot{Q}(x) + h(Pdx)T(x) = 0$$

Pdx is the area exposed to the surroundings for a given length dx where P is the perimeter. Dividing the equation by dx gives us:

$$\frac{\dot{Q}(x+dx) - \dot{Q}(x)}{dx} + hPT(x) = 0$$

Noting that $\frac{\dot{Q}(x+dx)-\dot{Q}(x)}{dx} = \dot{Q}'(x)$ as $dx \to 0$, the equation becomes:

$$\dot{Q}'(x) + hPT(x) = 0$$

Replacing $\dot{Q}(x)$ with $-kAT'(x)$ from Equation 4 gives:

$$-kAT''(x) + hPT(x) = 0$$

Rearranging and letting $\alpha = \sqrt{\frac{hP}{kA}}$ gives:

$$T''(x) - \alpha^2 T(x) = 0 \tag{5}$$

This will be the governing differential equation for both cases. Solving Equation 5 using the characteristic equation $r^2 - \alpha^2 = 0$ gives:

$$T(x) = Ae^{\alpha x} + Be^{-\alpha x}$$

Using the definitions $\sinh(\omega x) = \frac{e^{\omega x}-e^{-\omega x}}{2}$ and $\cosh(\omega x) = \frac{e^{\omega x}+e^{-\omega x}}{2}$, we get:

$$T(x) = C\sinh(\alpha x) + D\cosh(\alpha x) \tag{6}$$

Due to the interface in the bar, denoted at $\bar{x}$, the general solution is applied twice over each section with independent α_1 and α_2.

$$\begin{aligned} T_1(x) &= C_1 \sinh(\alpha_1 x) + D_1 \cosh(\alpha_1 x) \quad \text{for} \quad 0 \leq x \leq \bar{x} \\ T_2(x) &= C_2 \sinh(\alpha_2 x) + D_2 \cosh(\alpha_2 x) \quad \text{for} \quad \bar{x} \leq x \leq L \end{aligned} \tag{7}$$

The constants C_1, D_1, C_2, and D_2 must be solved simultaneously using the conditions that $T_1(0) = 0$, $T_1(L) = 100$, $T_1(\bar{x}) = T_2(\bar{x})$, and $-k_1 A_1 T_1'(\bar{x}) = k_2 A_2 T_2'(\bar{x})$.

1.3 Interface Temperature Results

1.3.1 Interface Location 1: $\bar{x} = 0.5$

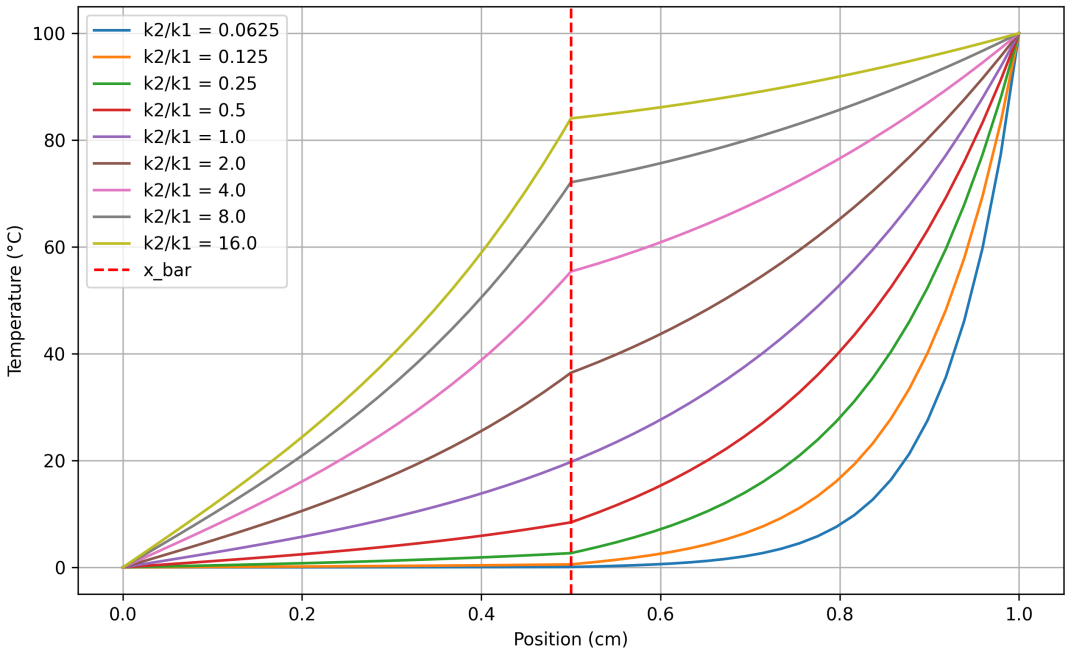

Figure 1: Analytical Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

Table 1: Analytical Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

x	$\frac{k_2}{k_1}$ 0.0625	0.125	0.25	0.5	1.0	2.0	4.0	8.0	16.0
0.000	0	0	0	0	0	0	0	0	0
0.125	0.0117	0.0976	0.4648	1.471	3.440	6.352	9.653	12.57	14.65
0.250	0.0252	0.2107	1.003	3.175	7.425	13.71	20.83	27.13	31.63
0.375	0.0427	0.3572	1.700	5.381	12.59	23.24	35.31	45.98	53.60
0.500	0.0669	0.5602	2.667	8.439	19.74	36.44	55.38	72.11	84.07
0.625	0.8478	3.304	8.627	17.44	30.02	45.82	62.46	76.70	86.74
0.750	4.228	10.63	20.26	32.03	45.04	58.79	71.99	82.80	90.26
0.875	20.57	32.67	45.24	56.89	67.20	76.39	84.34	90.51	94.67
1.000	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

1.3.2 Interface Location 2: $\bar{x} = \frac{2}{pi}$

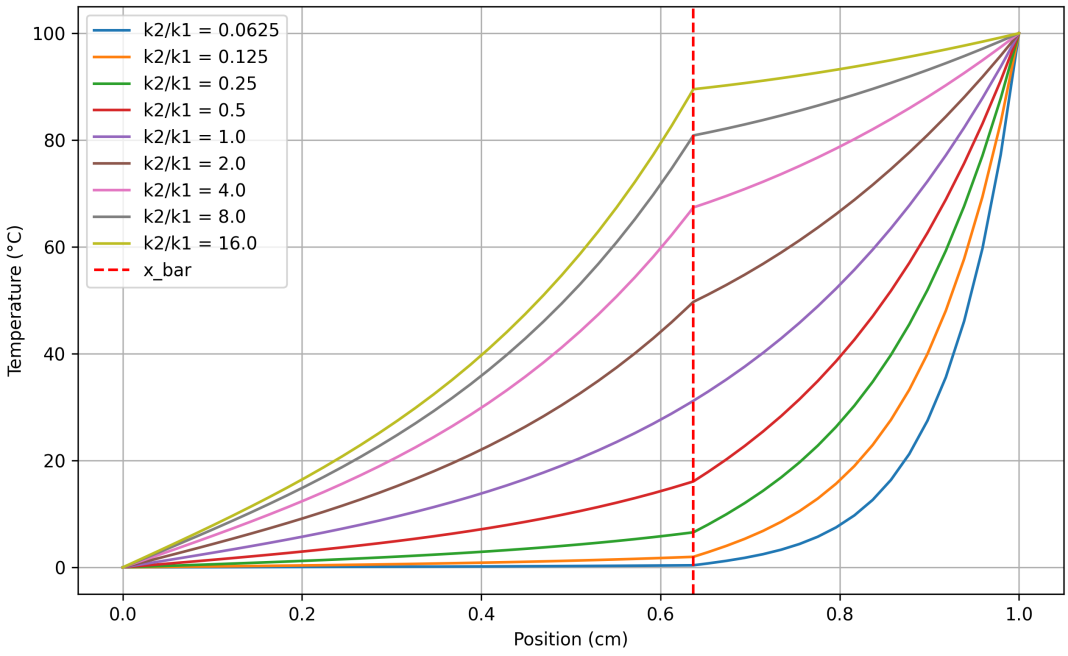

Figure 2: Analytical Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

Table 2: Analytical Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

x	$\frac{k_2}{k_1}$ 0.0625	0.125	0.25	0.5	1.0	2.0	4.0	8.0	16.0
0.000	0	0	0	0	0	0	0	0	0
0.125	0.0433	0.2178	0.7238	1.775	3.440	5.484	7.435	8.922	9.877
0.250	0.0934	0.4700	1.562	3.831	7.425	11.84	16.05	19.26	21.32
0.375	0.158	0.7967	2.648	6.494	12.59	20.06	27.20	32.64	36.13
0.500	0.248	1.249	4.153	10.18	19.74	31.46	42.65	51.19	56.67
0.625	0.377	1.900	6.315	15.49	30.02	47.85	64.86	77.84	86.18
0.637	0.392	1.974	6.561	16.09	31.18	49.71	67.38	80.86	89.53
0.750	4.086	10.00	18.93	30.68	45.04	60.65	74.76	85.28	91.97
0.875	20.55	32.49	44.74	56.31	67.20	77.28	85.70	91.74	95.52
1.000	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

1.3.3 Discussion

It can be seen from Figure 1 and Figure 2 that the section with lower thermal conductivity see a more gradual change in temperature compared to a higher k value. Additionally, comparing the two solutions with different interface locations shows a larger instantaneous temperature gradient at the interface.

1.4 Heat Convection Results

1.4.1 Interface Location 1: $\bar{x} = 0.5$

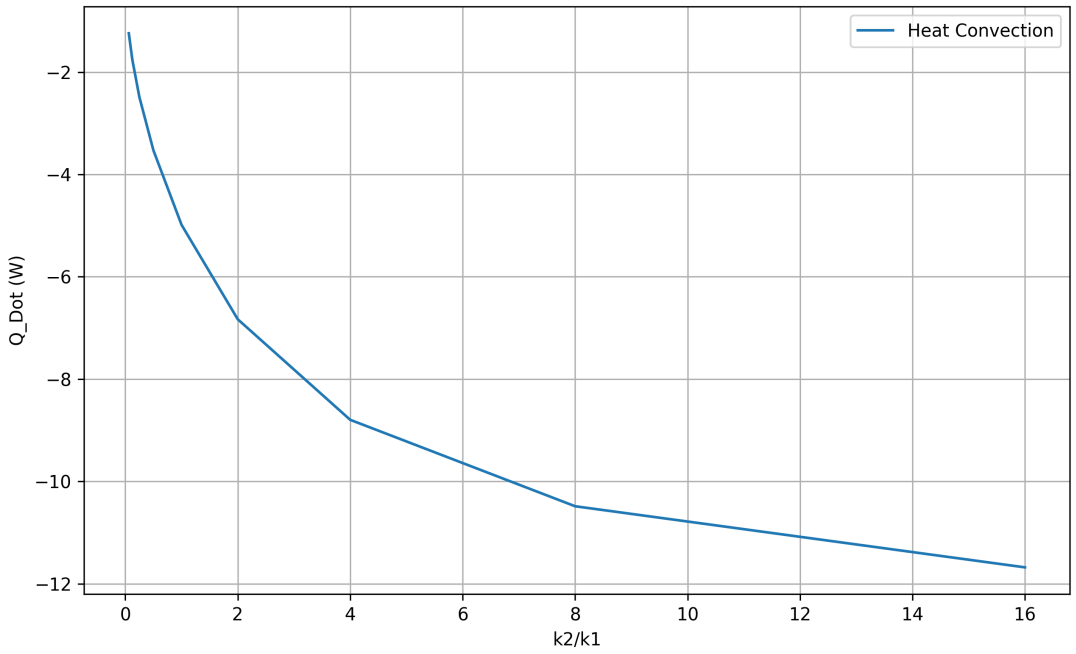

Figure 3: Analytical Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

Table 3: Analytical Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

$\frac{k_2}{k_1}$	Heat Convection
0.0625	-1.242
0.125	-1.756
0.25	-2.487
0.5	-3.529
1.0	-4.985
2.0	-6.832
4.0	-8.797
8.0	-10.49
16.0	-11.68

1.4.2 Interface Location 2: $\bar{x} = \frac{2}{pi}$

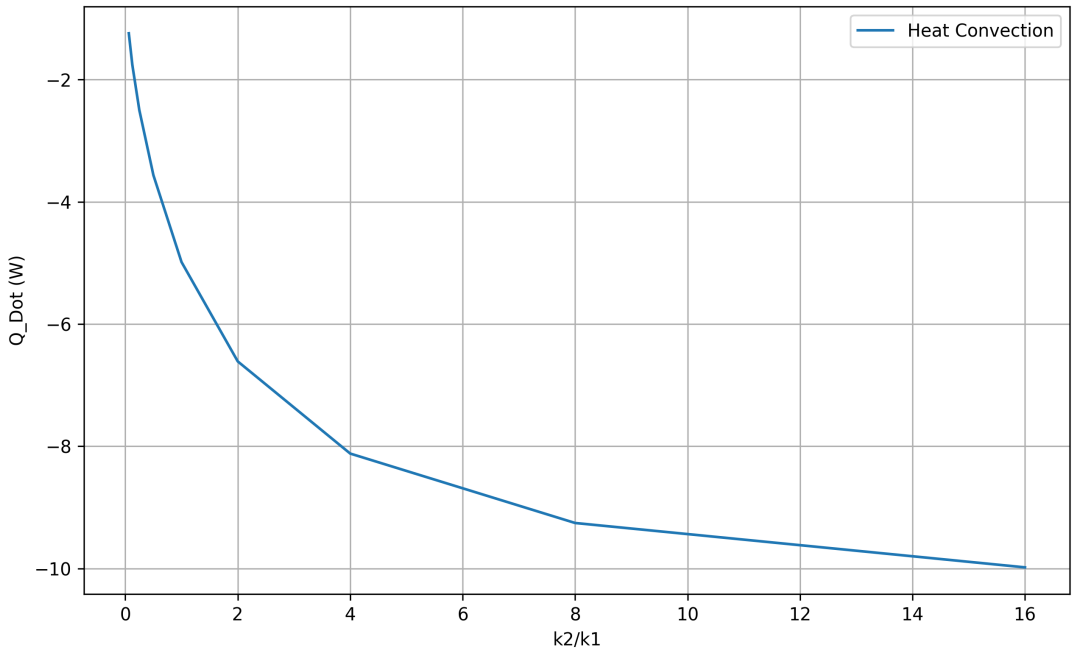

Figure 4: Analytical Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

Table 4: Analytical Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

$\frac{k_2}{k_1}$	Heat Convection
0.0625	-1.242
0.125	-1.759
0.25	-2.501
0.5	-3.564
1.0	-4.985
2.0	-6.611
4.0	-8.119
8.0	-9.255
16.0	-9.981

1.4.3 Discussion

From Figure 3 and Figure 4, there is larger heat loss due to convection when the thermal conductivity of the right section of the beam becomes larger.

2 FDM and FEM

2.1 Abstract

If the analytical solution to the temperature distribution is not know, the finite difference method (FDM) and or the finite element method (FEM) can be applied to the bimaterial beam to find the nodal temperatures.
While this report follows two sections of constant area and differing thermal conductivities, the FDM and FEM are generalized such that varying areas can be accounted for.

2.2 FDM Solution

Using Equation 4 and Equation 1 applied at each node, the following matrix equation can be setup for a mesh of $N = 6$ sections which can be easily extrapolated for any system of N sections.

$$\begin{bmatrix} k_1A_1w_1 & -k_1A_1 & 0 & 0 & 0 \\ -1 & k_1A_1w_1 & -k_1A_1 & 0 & 0 \\ 0 & -\beta_1 & \rho & -\beta_2 & 0 \\ 0 & 0 & -k_2A_2 & k_2A_2w_2 & -k_2A_2 \\ 0 & 0 & 0 & -k_2A_2 & k_2A_2w_2 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} T_1k_1A_1 = 0 \\ 0 \\ 0 \\ 0 \\ T_7k_2A_2 = 100k_2A_2 \end{Bmatrix} \tag{8}$$

Where:
$w_1 = 2 + \alpha_1^2\Delta x_1^2$
$w_2 = 2 + \alpha_2^2\Delta x_2^2$
$\beta_1 = \frac{k_1A_1}{\Delta x_1}$
$\beta_2 = \frac{k_2A_2}{\Delta x_2}$
$\rho = \beta_1 + \beta_2 + \frac{k_1A_1\Delta x_1\alpha_1^2}{2} + \frac{k_2A_2\Delta x_2\alpha_2^2}{2}$

2.3 FEM Solution

Using Equation 4 and Equation 1 applied at each node, the following matrix equation can be setup for a mesh of $N = 6$ sections which can be easily extrapolated for any system of N sections.

$$\begin{bmatrix} k_1A_1w_1 & -k_1A_1 & 0 & 0 & 0 \\ -1 & k_1A_1w_1 & -k_1A_1 & 0 & 0 \\ 0 & -\beta_1 & \rho & -\beta_2 & 0 \\ 0 & 0 & -k_2A_2 & k_2A_2w_2 & -k_2A_2 \\ 0 & 0 & 0 & -k_2A_2 & k_2A_2w_2 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} T_1k_1A_1 = 0 \\ 0 \\ 0 \\ 0 \\ T_7k_2A_2 = 100k_2A_2 \end{Bmatrix} \tag{9}$$

Where:

$w_1 = \frac{\frac{2}{\Delta x_1} + \frac{4\alpha_1^2\Delta x_1}{6}}{\frac{1}{\Delta x_1} - \frac{\alpha_1^2\Delta x_1}{6}}$

$w_2 = \frac{\frac{2}{\Delta x_2} + \frac{4\alpha_2^2\Delta x_2}{6}}{\frac{1}{\Delta x_2} - \frac{\alpha_1^2\Delta x_2}{6}}$

$\beta_1 = \frac{k_1A_1}{\Delta x_1}$

$\beta_2 = \frac{k_2A_2}{\Delta x_2}$

$\rho = \beta_1 + \beta_2 + \frac{k_1A_1\Delta x_1\alpha_1^2}{2} + \frac{k_2A_2\Delta x_2\alpha_2^2}{2}$

2.4 Interface Temperature Results

2.4.1 Interface Location 1: $\bar{x} = 0.5$

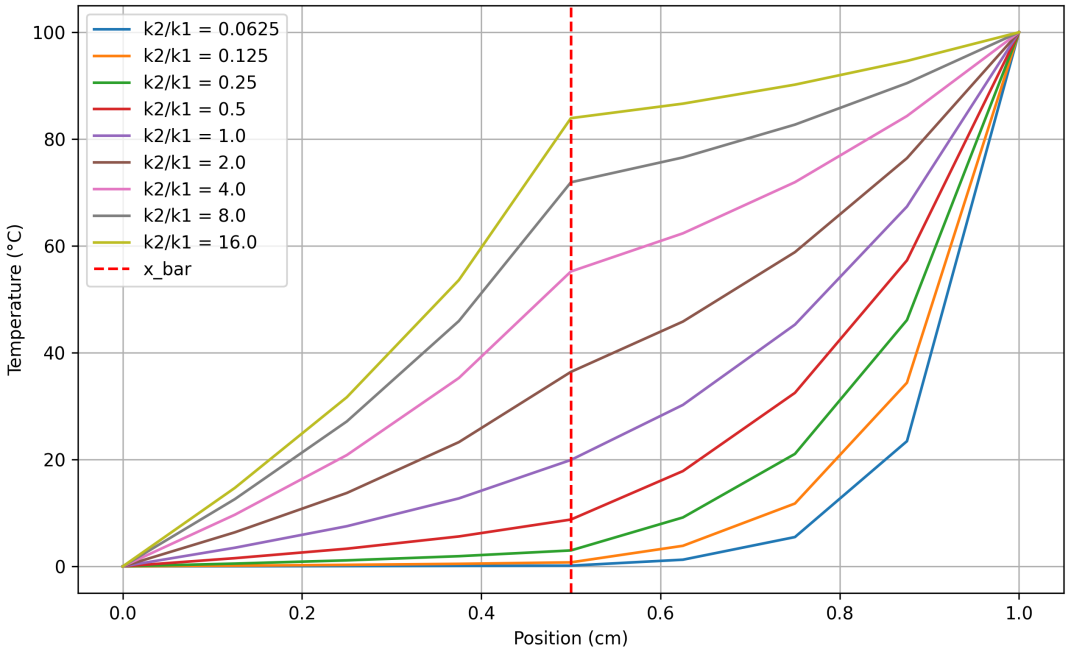

Figure 5: FDM Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

Table 5: FDM Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

x	$\frac{k_2}{k_1}$ 0.0625	0.125	0.25	0.5	1.0	2.0	4.0	8.0	16.0
0.000	0	0	0	0	0	0	0	0	0
0.125	0.0236	0.1312	0.5222	1.535	3.487	6.374	9.664	12.59	14.69
0.250	0.0508	0.2828	1.126	3.309	7.520	13.74	20.84	27.14	31.68
0.375	0.0860	0.4787	1.906	5.601	12.73	23.26	35.27	45.94	53.61
0.500	0.1346	0.7493	2.983	8.768	19.92	36.41	55.21	71.91	83.92
0.625	1.249	3.851	9.158	17.84	30.23	45.83	62.35	76.56	86.63
0.750	5.487	11.77	21.06	32.49	45.26	58.83	71.93	82.71	90.19
0.875	23.44	34.39	46.12	57.29	67.37	76.43	84.32	90.47	94.64
1.000	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

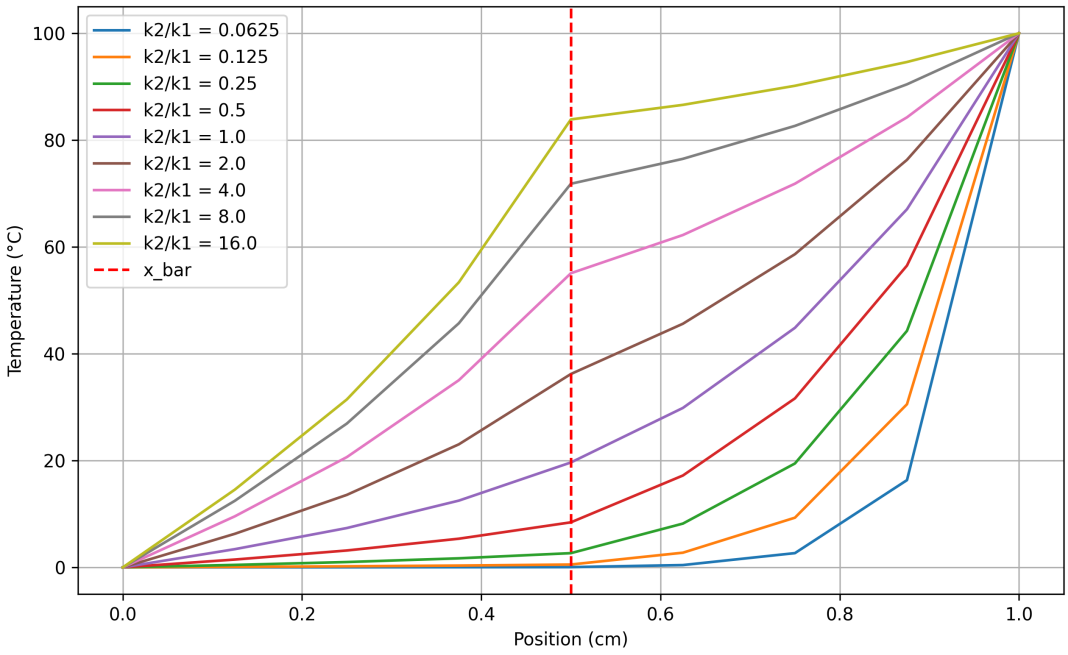

Figure 6: FEM Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

Table 6: FEM Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

x	$\frac{k_2}{k_1}$ 0.0625	0.125	0.25	0.5	1.0	2.0	4.0	8.0	16.0
0.000	0	0	0	0	0	0	0	0	0
0.125	0.00804	0.09242	0.4620	1.463	3.408	6.282	9.556	12.46	14.55
0.250	0.01737	0.1997	0.9980	3.161	7.363	13.57	20.65	26.93	31.44
0.375	0.02949	0.3389	1.694	5.366	12.50	23.04	35.05	45.71	53.37
0.500	0.04633	0.5326	2.662	8.432	19.64	36.20	55.07	71.82	83.87
0.625	0.4317	2.748	8.198	17.20	29.85	45.61	62.23	76.49	86.59
0.750	2.667	9.301	19.45	31.64	44.85	58.64	71.83	82.66	90.17
0.875	16.33	30.54	44.28	56.50	67.05	76.30	84.26	90.44	94.62
1.000	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

2.4.2 Interface Location 2: $\bar{x} = \frac{2}{pi}$

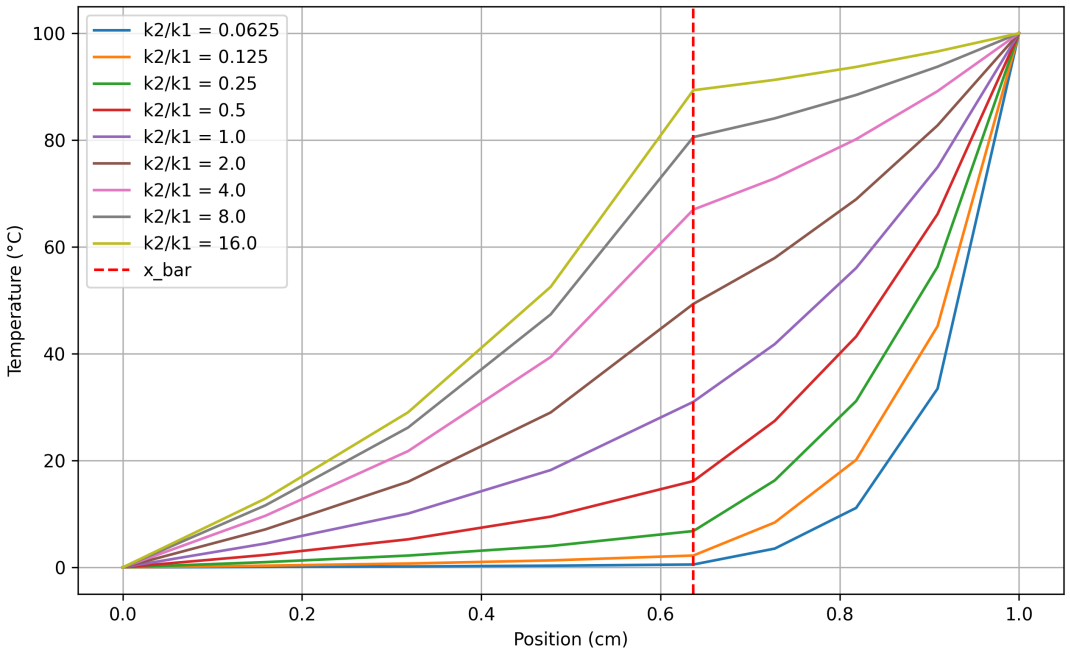

Figure 7: FDM Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

Table 7: FDM Temperature Distribution for Varying $\frac{k_2}{k_1}$ with $\bar{x} = \frac{2}{\pi}$

x	$\frac{k_2}{k_1}$ 0.0625	0.125	0.25	0.5	1.0	2.0	4.0	8.0	16.0
0.000	0	0	0	0	0	0	0	0	0
0.159	0.0771	0.3197	0.9805	2.330	4.469	7.113	9.660	11.62	12.88
0.318	0.1736	0.7204	2.209	5.250	10.07	16.03	21.77	26.18	29.03
0.477	0.3142	1.304	3.998	9.501	18.22	29.00	39.39	47.37	52.53
0.637	0.5343	2.217	6.799	16.16	30.99	49.33	66.99	80.56	89.33
0.727	3.516	8.402	16.28	27.43	41.80	57.93	72.83	84.07	91.28
0.818	11.14	20.13	31.14	43.22	56.05	68.92	80.17	88.45	93.70
0.909	33.47	45.16	56.28	66.15	74.93	82.75	89.16	93.74	96.60
1.000	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

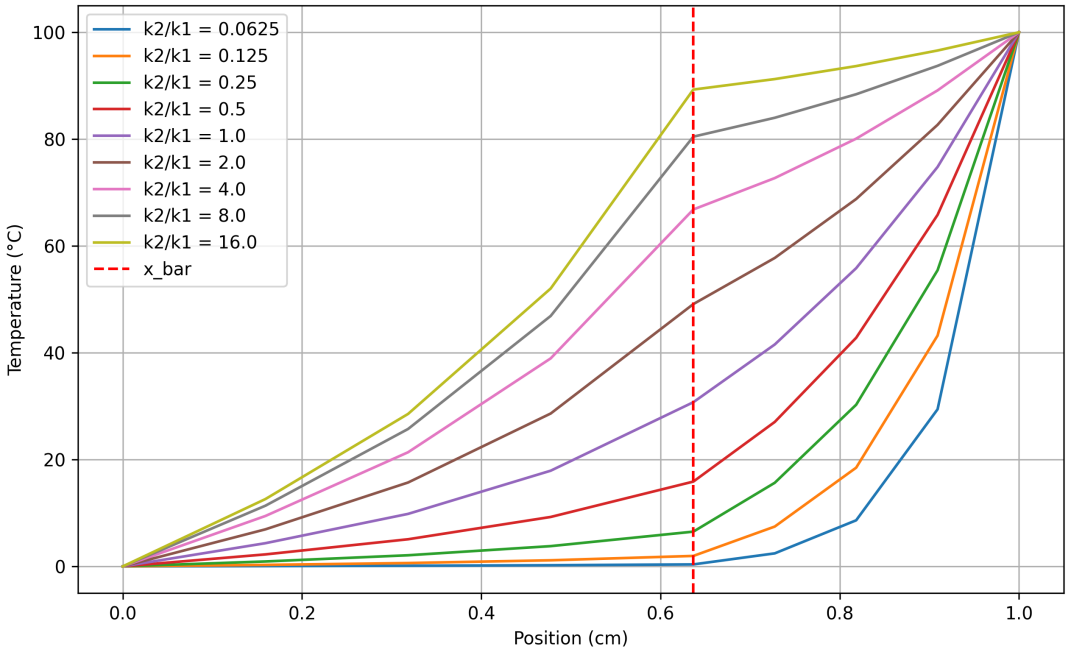

Figure 8: FEM Temperature Distribution for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

Table 8: FEM Temperature Distribution for Varying $\frac{k_2}{k_1}$ with $\bar{x} = \frac{2}{\pi}$

x	$\frac{k_2}{k_1}$ 0.0625	0.125	0.25	0.5	1.0	2.0	4.0	8.0	16.0
0.000	0	0	0	0	0	0	0	0	0
0.159	0.0519	0.2759	0.9180	2.241	4.339	6.933	9.434	11.36	12.60
0.318	0.1175	0.6247	2.078	5.075	9.826	15.70	21.36	25.72	28.54
0.477	0.2141	1.139	3.789	9.251	17.91	28.62	38.94	46.89	52.03
0.637	0.3674	1.954	6.502	15.87	30.73	49.11	66.82	80.45	89.27
0.727	2.435	7.452	15.65	27.04	41.54	57.74	72.70	83.99	91.23
0.818	8.624	18.48	30.26	42.79	55.83	68.78	80.08	88.39	93.67
0.909	29.41	43.21	55.45	65.81	74.79	82.67	89.12	93.71	96.58
1.000	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

2.4.3 Discussion

From the results in Figure 5, Figure 6, Figure 7, and Figure 8, both FDM and FEM showed similar results to the analytical temperature distribution in Figure 1 and Figure 2.

2.5 Heat Convection Results

2.5.1 Interface Location 1: $\bar{x} = 0.5$

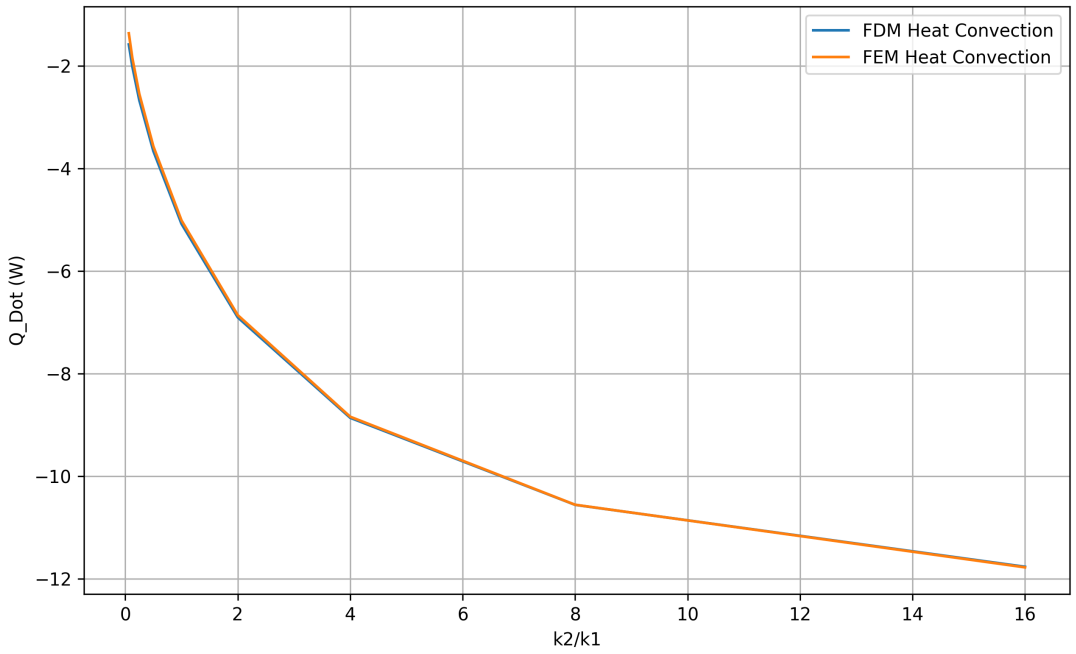

Figure 9: FDM and FEM Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

Table 9: FDM and FEM Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

$\frac{k_2}{k_1}$	FDM Q	FEM Q
0.0625	-1.58304	-1.36507
0.125	-2.01235	-1.84552
0.25	-2.67453	-2.54990
0.5	-3.66509	-3.57230
1.0	-5.08231	-5.01478
2.0	-6.90522	-6.86111
4.0	-8.86402	-8.84252
8.0	-10.5613	-10.5591
16.0	-11.7687	-11.7803

2.5.2 Interface Location 2: $\bar{x} = \frac{2}{pi}$

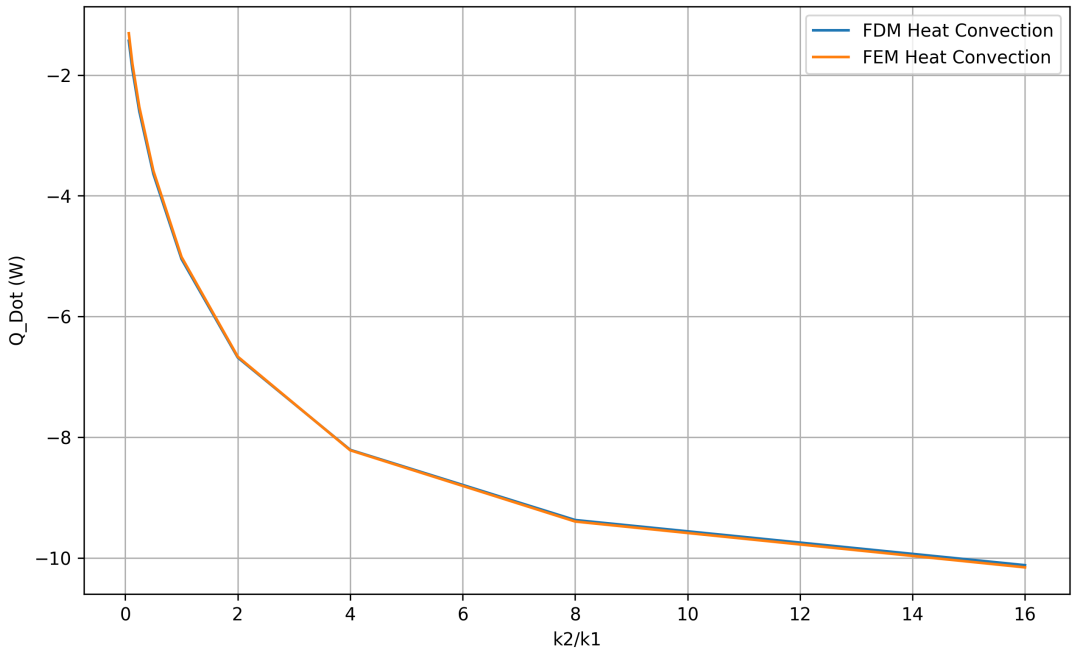

Figure 10: FDM and FEM Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

Table 10: FDM and FEM Heat Convection for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

$\frac{k_2}{k_1}$	FDM Q	FEM Q
0.0625	-1.43246	-1.30844
0.125	-1.89880	-1.80586
0.25	-2.60334	-2.53347
0.5	-3.63985	-3.58790
1.0	-5.04765	-5.01342
2.0	-6.67856	-6.66456
4.0	-8.20809	-8.21492
8.0	-9.37058	-9.39479
16.0	-10.1183	-10.1544

2.5.3 Discussion

Looking at the heat convection results in Figure 9 and Figure 10 compared to Figure 3 and Figure 4 show similar heat convection results at the end of the bar for both methods and interface locations.

3 FDM and FEM Convergence

3.1 Abstract

The convergence of the FDM and FEM are important quantities when verifying the method and predicting future performance for different setups. The rate of convergence of both methods will be calculated in reference to the exact quantity and the quantity found through Richardson Extrapolation.
The convergence of the interface temperature and the heat convection at the right end of the rod will be analyzed.

3.2 Interface Temperature Convergence

3.2.1 Interface Location 1: $\bar{x} = 0.5$

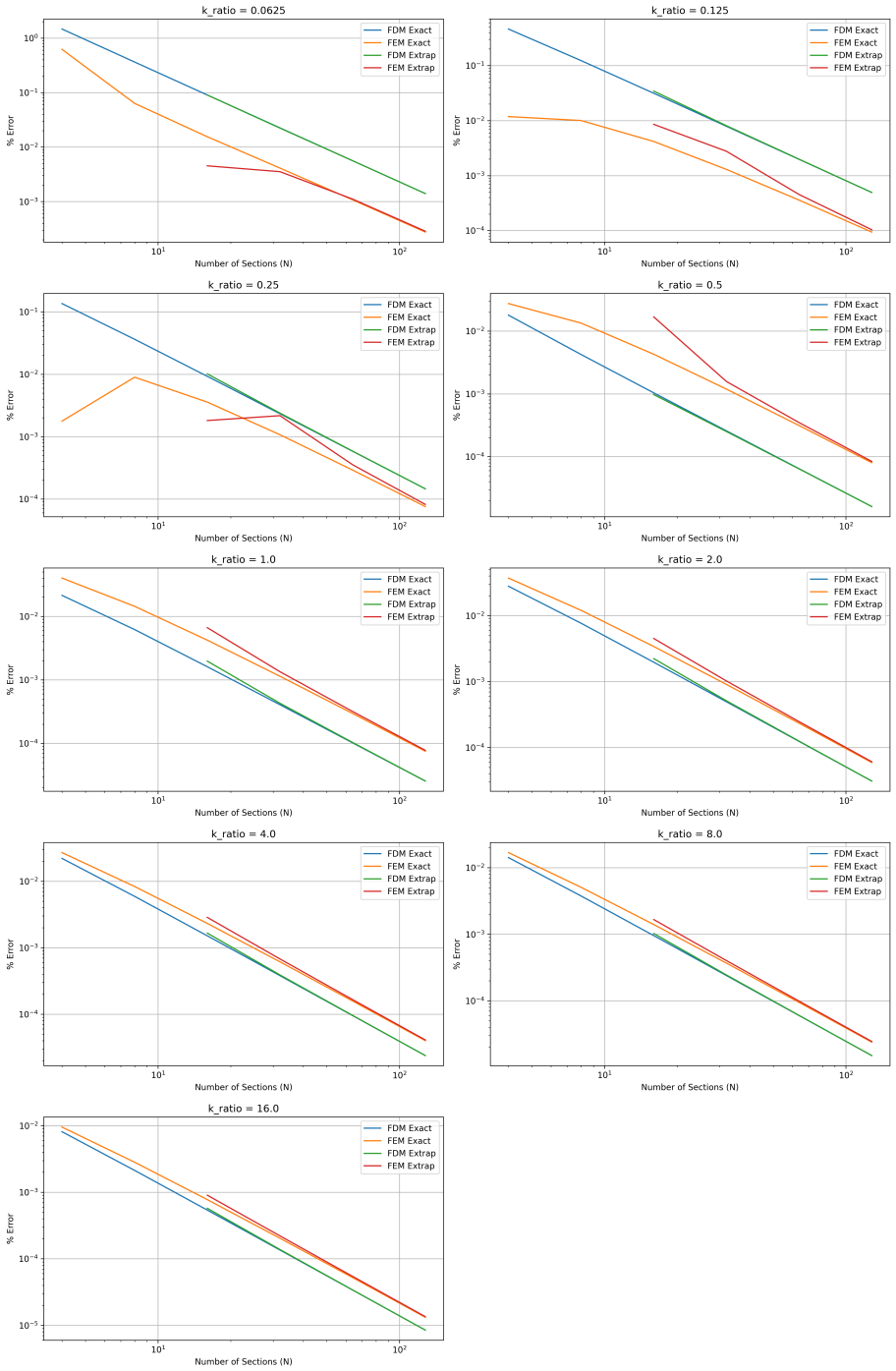

Figure 11: FDM and FEM Temperature Convergence for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

Table 11: FDM Temperature Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = 0.5$

N	True T	T	T_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	55.3837	54.7385	N/A	0.01165	N/A	N/A	N/A
8	55.3837	55.2113	N/A	0.003114	1.90385	N/A	N/A
16	55.3837	55.3399	55.3879	0.000792	1.97491	0.000867	1.87879
32	55.3837	55.3727	55.3840	0.000199	1.99366	0.000204	1.96857
64	55.3837	55.3810	55.3838	0.000050	1.99841	0.000050	1.99207
128	55.3837	55.3831	55.3837	0.000012	1.99960	0.000012	1.99801

Table 12: FEM Temperature Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = 0.5$

N	True T	T	T_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	55.3837	54.4102	N/A	0.01758	N/A	N/A	N/A
8	55.3837	55.0701	N/A	0.005663	1.63409	N/A	N/A
16	55.3837	55.2956	55.4126	0.001592	1.83095	0.002112	1.54918
32	55.3837	55.3604	55.3866	0.000421	1.91837	0.000472	1.79817
64	55.3837	55.3778	55.3841	0.000108	1.95986	0.000114	1.90373
128	55.3837	55.3822	55.3838	0.000027	1.98009	0.000028	1.95292

All other tables for their respective $\frac{k2}{k1}$ value are given in the Raw Code Output Appendix due to the sheer volume of data output.

3.2.2 Interface Location 2: $\bar{x} = \frac{2}{pi}$

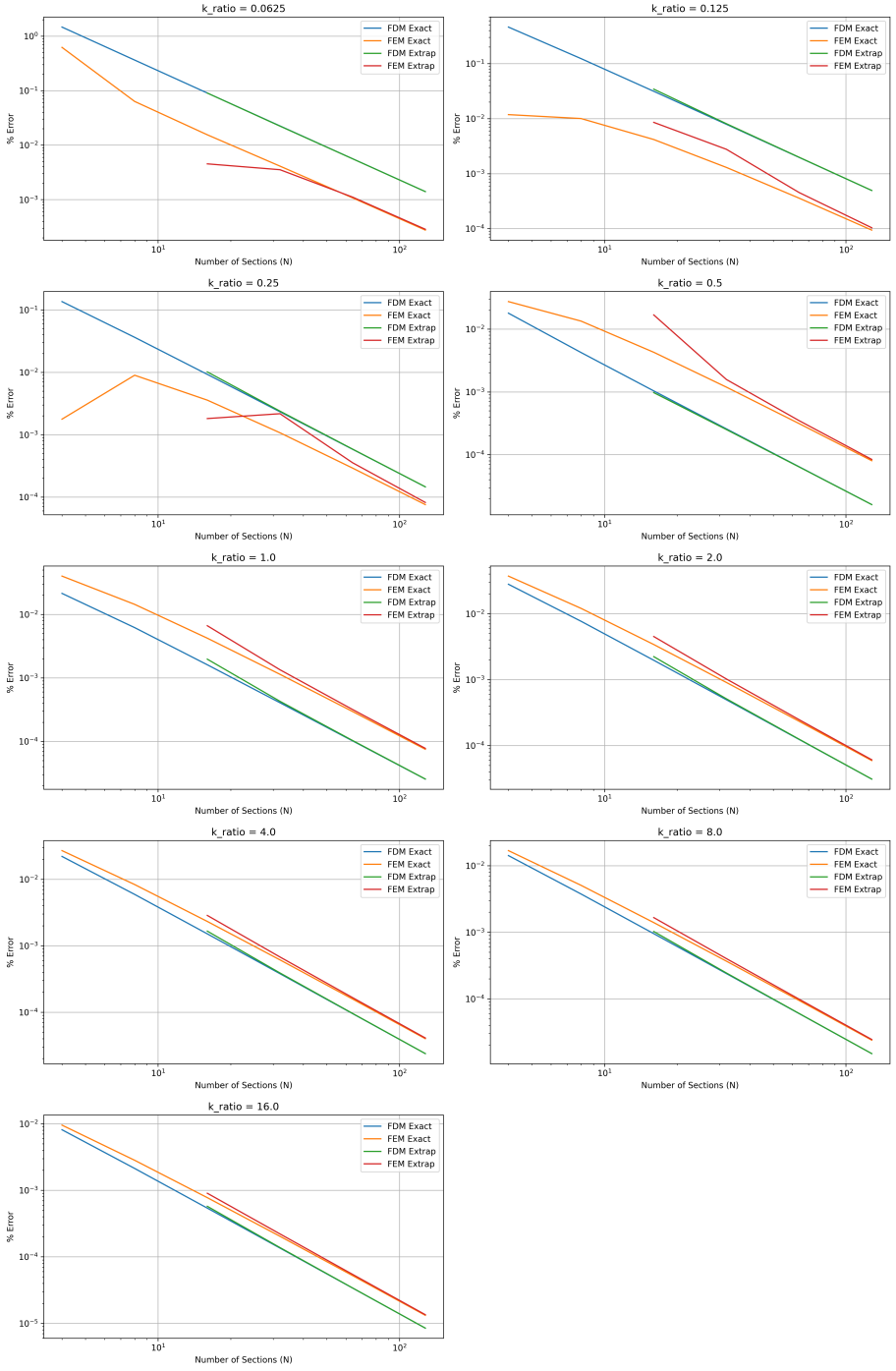

Figure 12: FDM and FEM Temperature Convergence for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

Table 13: FDM Temperature Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = \frac{2}{\pi}$

N	True T	T	T_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	67.3849	65.8951	N/A	0.022108	N/A	N/A	N/A
8	67.3849	66.9854	N/A	0.005928	1.89907	N/A	N/A
16	67.3849	67.2831	67.3949	0.001510	1.97291	0.001659	1.87294
32	67.3849	67.3593	67.3855	0.000379	1.99310	0.000389	1.96607
64	67.3849	67.3785	67.3849	0.000095	1.99827	0.000096	1.99137
128	67.3849	67.3833	67.3849	0.000024	1.99957	0.000024	1.99783

Table 14: FEM Temperature Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = \frac{2}{\pi}$

N	True T	T	T_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	67.3849	65.5624	N/A	0.027046	N/A	N/A	N/A
8	67.3849	66.8226	N/A	0.008344	1.69667	N/A	N/A
16	67.3849	67.2284	67.4212	0.002321	1.84583	0.002859	1.63480
32	67.3849	67.3436	67.3892	0.000613	1.92198	0.000677	1.81752
64	67.3849	67.3743	67.3854	0.000157	1.96071	0.000165	1.90834
128	67.3849	67.3822	67.3849	0.000040	1.98028	0.000041	1.95401

All other tables for their respective $\frac{k2}{k1}$ value are given in the Raw Code Output Appendix due to the sheer volume of data output.

3.2.3 Discussion

For both methods, the rate of convergence for the interface temperature compared to the analytical and extrapolated solution was $\beta = 2$. Notably, for smaller values of $\frac{k2}{k1}$, the FEM convergence was non linear until the number of sections N surpassed 32. This occurred with both interface locations. As seen from looking at the temperature distributions in Figure 1 and Figure 2, the non uniform convergence values correspond to sections of the bar with much larger temperature gradients and therefore heat flux.

3.3 Heat Convection Convergence

3.3.1 Interface Location 1: $\bar{x} = 0.5$

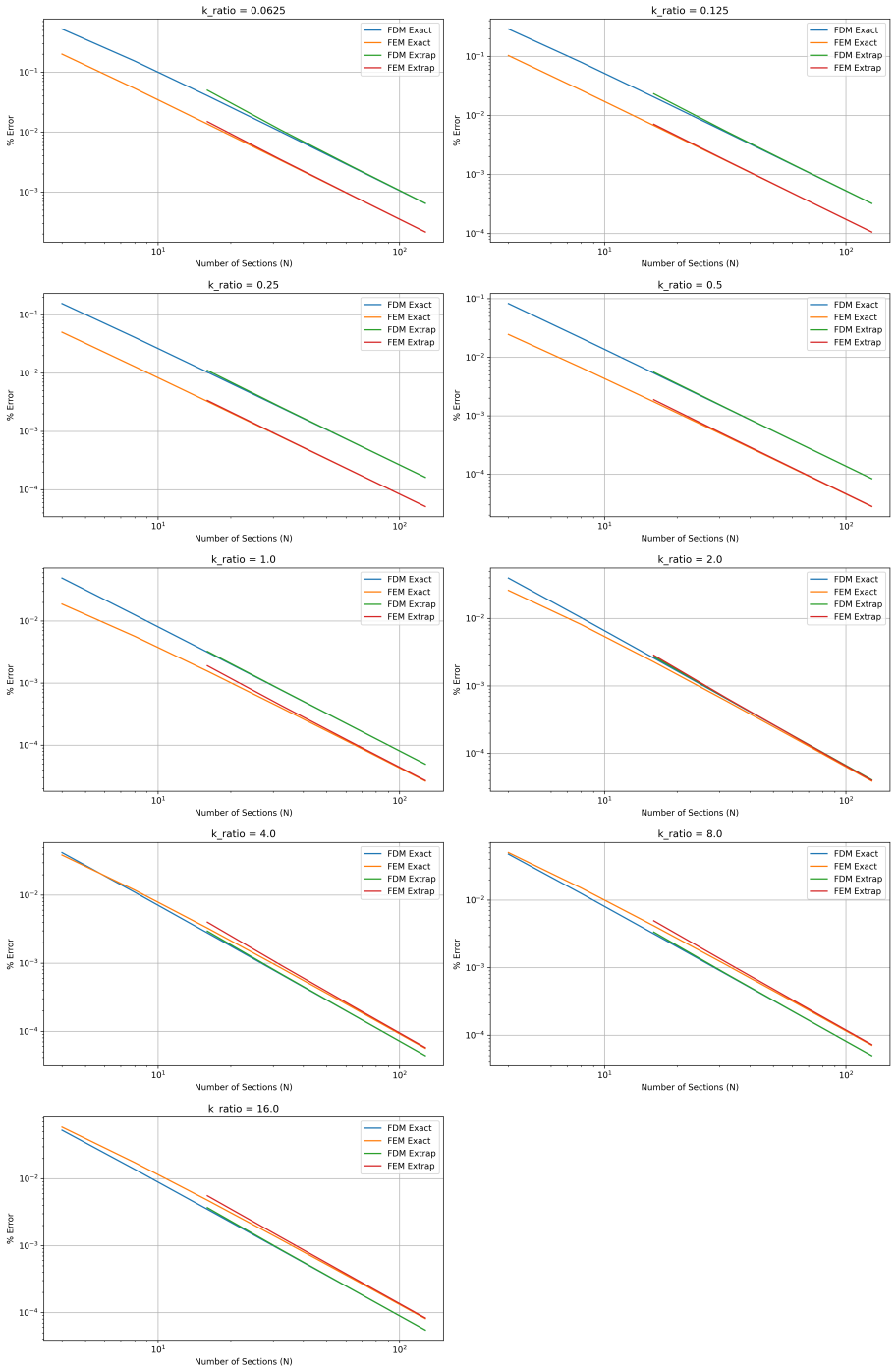

Figure 13: FDM and FEM Heat Convection Convergence for Varying $\frac{k2}{k1}$ with $\bar{x} = 0.5$

Table 15: FDM Heat Convection Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = 0.5$

N	True Q	Q	Q_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	-8.7970	-9.0603	N/A	0.029924	N/A	N/A	N/A
8	-8.7970	-8.8640	N/A	0.007612	1.97486	N/A	N/A
16	-8.7970	-8.8139	-8.7967	0.001912	1.99334	0.001956	1.96860
32	-8.7970	-8.8013	-8.7970	0.000479	1.99831	0.000481	1.99167
64	-8.7970	-8.7981	-8.7970	0.000120	1.99958	0.000120	1.99789
128	-8.7970	-8.7973	-8.7970	0.000030	1.99989	0.000030	1.99947

Table 16: FEM Heat Convection Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = 0.5$

N	True Q	Q	Q_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	-8.7970	-8.9466	N/A	0.017004	N/A	N/A	N/A
8	-8.7970	-8.8425	N/A	0.005169	1.7180	N/A	N/A
16	-8.7970	-8.8095	-8.7942	0.001419	1.8653	0.001740	1.6581
32	-8.7970	-8.8003	-8.7967	0.000371	1.9340	0.000406	1.8401
64	-8.7970	-8.7979	-8.7970	0.000095	1.9673	0.000099	1.9224
128	-8.7970	-8.7973	-8.7970	0.000024	1.9838	0.000025	1.9617

All other tables for their respective $\frac{k2}{k1}$ value are given in the Raw Code Output Appendix due to the sheer volume of data output.

3.3.2 Interface Location 2: $\bar{x} = \frac{2}{pi}$

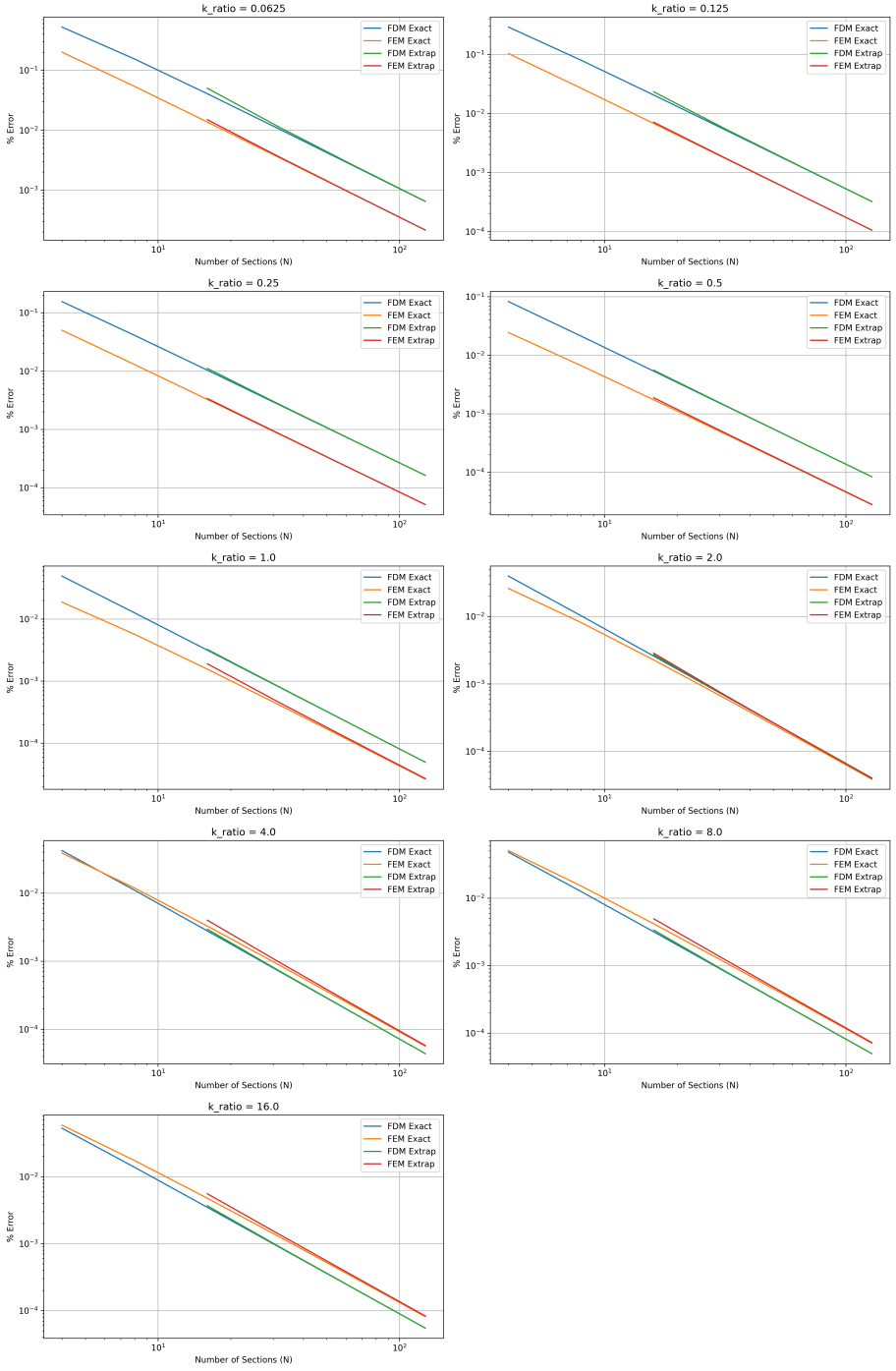

Figure 14: FDM and FEM Heat Convection Convergence for Varying $\frac{k2}{k1}$ with $\bar{x} = \frac{2}{\pi}$

Table 17: FDM Heat Convection Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = \frac{2}{\pi}$

N	True Q	Q	Q_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	-8.1187	-8.4608	N/A	0.042135	N/A	N/A	N/A
8	-8.1187	-8.2081	N/A	0.011009	1.93636	N/A	N/A
16	-8.1187	-8.1413	-8.1174	0.002786	1.98260	0.002953	1.92036
32	-8.1187	-8.1244	-8.1186	0.000699	1.99554	0.000710	1.97824
64	-8.1187	-8.1201	-8.1187	0.000175	1.99888	0.000175	1.99443
128	-8.1187	-8.1191	-8.1187	0.000044	1.99972	0.000044	1.99860

Table 18: FEM Heat Convection Convergence for $\frac{k2}{k1} = 4.0$ with $\bar{x} = \frac{2}{\pi}$

N	True Q	Q	Q_{EXTRAP}	$\%\text{Error}_{\text{EX}}$	β_{EX}	$\%\text{Error}_{\text{EXTRAP}}$	β_{EXTRAP}
4	-8.1187	-8.4335	N/A	0.038775	N/A	N/A	N/A
8	-8.1187	-8.2149	N/A	0.011850	1.71028	N/A	N/A
16	-8.1187	-8.1454	-8.1129	0.003283	1.85197	0.004001	1.65208
32	-8.1187	-8.1257	-8.1180	0.000864	1.92490	0.000951	1.82497
64	-8.1187	-8.1205	-8.1186	0.000222	1.96214	0.000233	1.91181
128	-8.1187	-8.1192	-8.1187	0.000056	1.98099	0.000058	1.95569

All other tables for their respective $\frac{k2}{k1}$ value are given in the Raw Code Output Appendix due to the sheer volume of data output.

3.3.3 Discussion

For both methods, the rate of convergence for the heat convection compared to the analytical and extrapolated solution was $\beta = 2$. Interestingly, the non linear behavior as seen in the temperature convergence is not reflected here.

4 Raw Code Output

```
Section 1: Analytical
Analytical Temperature Results x_bar = 0.5:
              x = 0.000  x = 0.125  x = 0.250  x = 0.375  x = 0.500  x = 0.500  x = 0.625  x = 0.750  x = 0.875  x = 1.000
k2/k1 = 0.0625        0.0   0.011667   0.025181   0.042681   0.066937   0.066937   0.847794   4.228179  20.573130      100.0
k2/k1 = 0.125         0.0   0.097641   0.210739   0.357195   0.560194   0.560194   3.303981  10.626282  32.674022      100.0
k2/k1 = 0.25          0.0   0.464798   1.003171   1.700341   2.666666   2.666666   8.626608  20.264910  45.242343      100.0
k2/k1 = 0.5           0.0   1.470842   3.174511   5.380689   8.438605   8.438605  17.437015  32.027881  56.890833      100.0
k2/k1 = 1.0           0.0   3.440413   7.425426  12.585848  19.738549  19.738549  30.015765  45.044332  67.203196      100.0
k2/k1 = 2.0           0.0   6.351689  13.708819  23.235988  36.441301  36.441301  45.815586  58.792577  76.392720      100.0
k2/k1 = 4.0           0.0   9.653341  20.834759  35.314219  55.383747  55.383747  62.464754  71.993744  84.344156      100.0
k2/k1 = 8.0           0.0  12.567937  27.125318  45.976504  72.105546  72.105546  76.701212  82.797389  90.513336      100.0
k2/k1 = 16.0          0.0  14.652807  31.625083  53.603455  84.066993  84.066993  86.741990  90.264767  94.669752      100.0

Analytical Temperature Results x_bar = 2/pi:
              x = 0.000  x = 0.125  x = 0.250  x = 0.375  x = 0.500  x = 0.625  x = 0.637  x = 0.750  x = 0.875  x = 1.000
k2/k1 = 0.0625        0.0   0.043265   0.093379   0.158275   0.248224   0.377466   0.392138   4.086217  20.545109      100.0
k2/k1 = 0.125         0.0   0.217770   0.470012   0.796654   1.249403   1.899926   1.973771  10.004385  32.490342      100.0
k2/k1 = 0.25          0.0   0.723845   1.562272   2.647999   4.152891   6.315166   6.560619  18.929061  44.739811      100.0
k2/k1 = 0.5           0.0   1.775153   3.831304   6.493931  10.184517  15.487261  16.089210  30.677503  56.308955      100.0
k2/k1 = 1.0           0.0   3.440413   7.425426  12.585848  19.738549  30.015765  31.182398  45.044332  67.203196      100.0
k2/k1 = 2.0           0.0   5.484175  11.836467  20.062415  31.464146  47.846498  49.706164  60.645947  77.284348      100.0
k2/k1 = 4.0           0.0   7.434698  16.046273  27.197894  42.654810  64.863775  67.384857  74.763593  85.702464      100.0
k2/k1 = 8.0           0.0   8.921798  19.255874  32.638061  51.186694  77.837932  80.863285  85.279652  91.742445      100.0
k2/k1 = 16.0          0.0   9.877475  21.318505  36.134152  56.669658  86.175696  89.525116  91.966805  95.516633      100.0

Analytical Heat Convection Results x_bar = 0.5:
              Heat Convection
k2/k1 = 0.0625       -1.241829
k2/k1 = 0.125        -1.756437
k2/k1 = 0.25         -2.486946
k2/k1 = 0.5          -3.529489
k2/k1 = 1.0          -4.985127
k2/k1 = 2.0          -6.832006
k2/k1 = 4.0          -8.797049
k2/k1 = 8.0         -10.486214
k2/k1 = 16.0        -11.680625

Analytical Heat Convection Results x_bar = 2/pi:
              Heat Convection
k2/k1 = 0.0625       -1.241978
k2/k1 = 0.125        -1.758799
k2/k1 = 0.25         -2.501202
k2/k1 = 0.5          -3.564213
k2/k1 = 1.0          -4.985127
k2/k1 = 2.0          -6.610807
k2/k1 = 4.0          -8.118713
k2/k1 = 8.0          -9.254592
k2/k1 = 16.0         -9.980639

Section 2: FDM and FEM with Varying k_ratio
FDM Temperature Results x_bar = 0.5:
              x = 0.000  x = 0.125  x = 0.250  x = 0.375  x = 0.500  x = 0.625  x = 0.750  x = 0.875  x = 1.000
k2/k1 = 0.0625        0.0   0.023566   0.050814   0.086002   0.134628   1.249213   5.486829  23.441518      100.0
k2/k1 = 0.125         0.0   0.131160   0.282813   0.478656   0.749289   3.850963  11.766340  34.389643      100.0
k2/k1 = 0.25          0.0   0.522237   1.126073   1.905858   2.983433   9.158380  21.057315  46.117073      100.0
k2/k1 = 0.5           0.0   1.534785   3.309379   5.601064   8.767916  17.841592  32.490766  57.293304      100.0
k2/k1 = 1.0           0.0   3.487383   7.519670  12.726906  19.922720  30.231460  45.263865  67.368749      100.0
k2/k1 = 2.0           0.0   6.373779  13.743460  23.260557  36.412116  45.832593  58.833740  76.431273      100.0
k2/k1 = 4.0           0.0   9.664493  20.839062  35.269736  55.211305  62.353389  71.931152  84.318726      100.0
k2/k1 = 8.0           0.0  12.587755  27.142347  45.937931  71.911316  76.562507  82.709060  90.471024      100.0
k2/k1 = 16.0          0.0  14.689868  31.675027  53.609410  83.920263  86.634225  90.194225  94.635027      100.0

FEM Temperature Results x_bar = 0.5:
              x = 0.000  x = 0.125  x = 0.250  x = 0.375  x = 0.500  x = 0.625  x = 0.750  x = 0.875  x = 1.000
k2/k1 = 0.0625        0.0   0.008040   0.017369   0.029485   0.046331   0.431694   2.667176  16.333414      100.0
k2/k1 = 0.125         0.0   0.092421   0.199668   0.338947   0.532604   2.747608   9.300941  30.539969      100.0
k2/k1 = 0.25          0.0   0.461952   0.998014   1.694185   2.662150   8.197855  19.452994  44.279989      100.0
k2/k1 = 0.5           0.0   1.463251   3.161248   5.366397   8.432466  17.199749  31.637279  56.504681      100.0
k2/k1 = 1.0           0.0   3.408346   7.363486  12.499933  19.641717  29.852520  44.852497  67.048062      100.0
k2/k1 = 2.0           0.0   6.282365  13.572595  23.040247  36.204196  45.614623  58.635706  76.298138      100.0
k2/k1 = 4.0           0.0   9.556088  20.645239  35.046460  55.070107  62.227195  71.830962  84.259013      100.0
k2/k1 = 8.0           0.0  12.463187  26.925816  45.708094  71.823222  76.490410  82.656431  90.442107      100.0
k2/k1 = 16.0          0.0  14.553527  31.441845  53.374310  83.869497  86.594484  90.166500  94.620483      100.0

FDM Temperature Results x_bar = 2/pi:
              x = 0.000  x = 0.159  x = 0.318  x = 0.477  x = 0.637  x = 0.727  x = 0.818  x = 0.909  x = 1.000
k2/k1 = 0.0625        0.0   0.077055   0.173629   0.314183   0.534322   3.515595  11.139042  33.471059      100.0
k2/k1 = 0.125         0.0   0.319719   0.720423   1.303613   2.217011   8.401972  20.134133  45.159372      100.0
k2/k1 = 0.25          0.0   0.980497   2.209358   3.997855   6.799021  16.282888  31.141948  56.281369      100.0
k2/k1 = 0.5           0.0   2.330072   5.250358   9.500576  16.157318  27.426136  43.221817  66.151538      100.0
k2/k1 = 1.0           0.0   4.469024  10.070065  18.221884  30.989359  41.796025  56.052044  74.933939      100.0
k2/k1 = 2.0           0.0   7.113289  16.028395  29.003540  49.325368  57.925784  68.916457  82.750907      100.0
k2/k1 = 4.0           0.0   9.660074  21.767073  39.387737  66.985431  72.825251  80.167607  89.163985      100.0
k2/k1 = 8.0           0.0  11.617645  26.178075  47.369488  80.559732  84.071349  88.450249  93.741604      100.0
k2/k1 = 16.0          0.0  12.883057  29.029430  52.529043  89.334419  91.281472  93.699357  96.600545      100.0

FEM Temperature Results x_bar = 2/pi:
              x = 0.000  x = 0.159  x = 0.318  x = 0.477  x = 0.637  x = 0.727  x = 0.818  x = 0.909  x = 1.000
k2/k1 = 0.0625        0.0   0.051867   0.117450   0.214096   0.367363   2.434575   8.623646  29.412987      100.0
k2/k1 = 0.125         0.0   0.275874   0.624708   1.138757   1.953971   7.451625  18.477330  43.210596      100.0
k2/k1 = 0.25          0.0   0.917965   2.078702   3.789190   6.501798  15.648464  30.261653  55.446231      100.0
k2/k1 = 0.5           0.0   2.241155   5.075024   9.251075  15.873740  27.039252  42.794017  65.812018      100.0
k2/k1 = 1.0           0.0   4.339198   9.825975  17.911409  30.733838  41.542855  55.828147  74.785094      100.0
k2/k1 = 2.0           0.0   6.933484  15.700652  28.620142  49.108745  57.744459  68.779447  82.672214      100.0
k2/k1 = 4.0           0.0   9.434441  21.363990  38.943632  66.822621  72.697960  80.078383  89.116689      100.0
k2/k1 = 8.0           0.0  11.358900  25.721867  46.887445  80.453254  83.990156  88.394995  93.713291      100.0
k2/k1 = 16.0          0.0  12.604001  28.541357  52.026991  89.272101  91.234400  93.667692  96.584539      100.0

FDM Heat Convection Results x_bar = 0.5:
              Heat Convection
k2/k1 = 0.0625       -1.583037
k2/k1 = 0.125        -2.012353
k2/k1 = 0.25         -2.674530
k2/k1 = 0.5          -3.665089
k2/k1 = 1.0          -5.082312
k2/k1 = 2.0          -6.905215
k2/k1 = 4.0          -8.864016
k2/k1 = 8.0         -10.561319
k2/k1 = 16.0        -11.768665

FEM Heat Convection Results x_bar = 0.5:
              Heat Convection
k2/k1 = 0.0625       -1.365065
k2/k1 = 0.125        -1.845516
k2/k1 = 0.25         -2.549900
k2/k1 = 0.5          -3.572301
k2/k1 = 1.0          -5.014775
k2/k1 = 2.0          -6.861111
k2/k1 = 4.0          -8.842519
k2/k1 = 8.0         -10.559111
k2/k1 = 16.0        -11.780305

FDM Heat Convection Results x_bar = 2/pi:
              Heat Convection
k2/k1 = 0.0625       -1.432463
k2/k1 = 0.125        -1.898803
k2/k1 = 0.25         -2.603336
k2/k1 = 0.5          -3.639854
k2/k1 = 1.0          -5.047653
k2/k1 = 2.0          -6.678556
k2/k1 = 4.0          -8.208090
k2/k1 = 8.0          -9.370579
k2/k1 = 16.0        -10.118258

FEM Heat Convection Results x_bar = 2/pi:
              Heat Convection
k2/k1 = 0.0625       -1.308439
k2/k1 = 0.125        -1.805860
```

```
k2/k1 = 0.25          -2.533474
k2/k1 = 0.5           -3.587897
k2/k1 = 1.0           -5.013420
k2/k1 = 2.0           -6.664558
k2/k1 = 4.0           -8.214918
k2/k1 = 8.0           -9.394792
k2/k1 = 16.0         -10.154415


Section 3: Convergence of FDM and FEM with Varying k Ratio
Convergence of FDM and FEM Temperature at x_bar = 0.500 and k_ratio = 0.0625
          True T     FDM T  FDM Extrap T  FDM Exact  % Error  FDM Exact B  FDM Extrap  % Error  FDM Extrap B     FEM T  FEM Extrap T  FEM Exact % Error  FEM Exact B  FEM Extrap % Error  FEM Extrap B
N = 4    0.066937  0.400229           NaN   4.979197      NaN          NaN         NaN      NaN -0.363773          NaN          6.434572          NaN                 NaN           NaN
N = 8    0.066937  0.134628           NaN   1.011272  2.299743          NaN         NaN      NaN  0.046331          NaN          0.307843  4.385579                 NaN           NaN
N = 16   0.066937  0.082453       0.06969   0.231796  2.125242     0.183010    2.347809  0.062673  0.063352          0.063696  2.272928             0.010706      4.649294
N = 32   0.066937  0.070718       0.067312  0.056482  2.036995     0.050588    2.152561  0.065888  0.066675          0.015670  2.023150             0.011806      2.345888
N = 64   0.066937  0.067876       0.066968  0.014026  2.009682     0.013561    2.045906  0.066671  0.066924          0.003966  1.982240             0.003772      2.036753
N = 128  0.066937  0.067171       0.066939  0.003501  2.002449     0.003469    2.012080  0.066870  0.066937          0.001003  1.984061             0.001005      1.981623

Convergence of FDM and FEM Heat Convection with x_bar = 0.500 and k_ratio = 0.0625
          True Q     FDM Q  FDM Extrap Q  FDM Exact  % Error FDM Exact B  FDM Extrap  % Error  FDM Extrap B     FEM Q  FEM Extrap Q  FEM Exact % Error FEM Exact B  FEM Extrap % Error  FEM Extrap B
N = 4   -1.241829 -2.323339           NaN   0.870901      [None]          NaN         NaN      NaN -1.681631          NaN          0.354157      [None]                 NaN           NaN
N = 8   -1.241829 -1.583037           NaN   0.274763  1.664323          NaN         NaN      NaN -1.365065          NaN          0.099238  1.835423                   NaN           NaN
N = 16  -1.241829 -1.335328    -1.210764    0.075292  1.867617     0.102880    1.579472 -1.273756 -1.236742          -1.274156 0.025710  1.948582             0.029928      1.793664
N = 32  -1.241829 -1.265851    -1.238768    0.019344  1.960577     0.021863    1.834029 -1.249887 -1.241439          -1.241827 0.006489  1.98625              0.006805      1.935639
N = 64  -1.241829 -1.247878    -1.241605    0.004871  1.989623     0.005052    1.950668 -1.243848 -1.241803          -1.241827 0.001626  1.9965               0.001647      1.982806
N = 128 -1.241829 -1.243344    -1.241814    0.001220  1.997371     0.001232    1.987025 -1.242334 -1.241827          -1.241827 0.000407  1.999121             0.000408      1.995625

Convergence of FDM and FEM Temperature at x_bar = 0.500 and k_ratio = 0.125
          True T     FDM T  FDM Extrap T  FDM Exact  % Error  FDM Exact B  FDM Extrap  % Error  FDM Extrap B     FEM T  FEM Extrap T  FEM Exact % Error  FEM Exact B  FEM Extrap % Error  FEM Extrap B
N = 4    0.560194  1.321586           NaN   1.359157      NaN          NaN         NaN      NaN  0.284007          NaN          0.493020          NaN                 NaN           NaN
N = 8    0.560194  0.749289           NaN   0.337552  2.009531          NaN         NaN      NaN  0.532604          NaN          0.049252  3.323401                 NaN           NaN
N = 16   0.560194  0.607201       0.560273  0.083911  2.008170     0.083760    2.009981  0.553343  0.555230          0.012231  2.009667             0.003400      3.583391
N = 32   0.560194  0.571925       0.560274  0.020940  2.002601     0.020794    2.010016  0.558340  0.559926          0.003310  1.885500             0.002833      2.053153
N = 64   0.560194  0.563125       0.560201  0.005233  2.000690     0.005220    2.003237  0.559704  0.560217          0.000875  1.920215             0.000915      1.872828
N = 128  0.560194  0.560927       0.560195  0.001308  2.000175     0.001307    2.000862  0.560068  0.560200          0.000226  1.955459             0.000236      1.907766

Convergence of FDM and FEM Heat Convection with x_bar = 0.500 and k_ratio = 0.125
          True Q     FDM Q  FDM Extrap Q  FDM Exact  % Error FDM Exact B  FDM Extrap  % Error  FDM Extrap B     FEM Q  FEM Extrap Q  FEM Exact % Error FEM Exact B  FEM Extrap % Error  FEM Extrap B
N = 4   -1.756437 -2.635211           NaN   0.500316      [None]          NaN         NaN      NaN -2.090293          NaN          0.190075      [None]                 NaN           NaN
N = 8   -1.756437 -2.012353           NaN   0.145701  1.779824          NaN         NaN      NaN -1.845516          NaN          0.050716  1.906073                   NaN           NaN
N = 16  -1.756437 -1.823780    -1.741900    0.038341  1.926064     0.047006    1.723786 -1.779124 -1.754414          -1.756295 0.012916  1.973249             0.014085      1.882382
N = 32  -1.756437 -1.773512    -1.755242    0.009721  1.97963      0.010409    1.907407 -1.762136 -1.756295          -1.756437 0.003245  1.993049             0.003326      1.966545
N = 64  -1.756437 -1.760722    -1.756356    0.002439  1.99477      0.002486    1.974522 -1.757864 -1.756428          -1.756437 0.000812  1.998239             0.000817      1.991312
N = 128 -1.756437 -1.757509    -1.756432    0.000610  1.998684     0.000613    1.993462 -1.756794 -1.756437          -1.756437 0.000203  1.999555             0.000203      1.997799

Convergence of FDM and FEM Temperature at x_bar = 0.500 and k_ratio = 0.25
          True T     FDM T  FDM Extrap T  FDM Exact  % Error  FDM Exact B  FDM Extrap  % Error  FDM Extrap B     FEM T  FEM Extrap T  FEM Exact % Error  FEM Exact B  FEM Extrap % Error  FEM Extrap B
N = 4    2.666666  3.862069           NaN   0.448276      NaN          NaN         NaN      NaN  2.715719          NaN          0.018395  NaN                      NaN           NaN
N = 8    2.666666  2.983433           NaN   0.118784  1.916002          NaN         NaN      NaN  2.662150          NaN          0.001694  3.441210                 NaN           NaN
N = 16   2.666666  2.747136       2.660209  0.030176  1.976907     0.032677    1.894660  2.661848  2.661847          0.001807  -0.093302            6.420173e-07  7.472074
N = 32   2.666666  2.686866       2.666231  0.007575  1.994067     0.007739    1.971109  2.664898  2.662123          0.000663  1.446493             1.042592e-03  -3.337586
N = 64   2.666666  2.671721       2.666638  0.001896  1.998506     0.001906    1.992582  2.666147  2.667012          0.000195  1.767683             3.244563e-04  1.288577
N = 128  2.666666  2.667930       2.666664  0.000474  1.999626     0.000475    1.998132  2.666526  2.666692          0.000052  1.891986             6.204964e-05  1.718994

Convergence of FDM and FEM Heat Convection with x_bar = 0.500 and k_ratio = 0.25
          True Q     FDM Q  FDM Extrap Q  FDM Exact  % Error FDM Exact B  FDM Extrap  % Error  FDM Extrap B     FEM Q  FEM Extrap Q  FEM Exact % Error FEM Exact B  FEM Extrap % Error  FEM Extrap B
N = 4   -2.486946 -3.171745           NaN   0.275357      [None]          NaN         NaN      NaN -2.730060          NaN          0.097756  [None]                 NaN           NaN
N = 8   -2.486946 -2.674530           NaN   0.075427  1.868146          NaN         NaN      NaN -2.549900          NaN          0.025314  1.949262               NaN           NaN
N = 16  -2.486946 -2.535136    -2.480834    0.019377  1.960724     0.021889    1.834707 -2.502844 -2.486207          -2.486894 0.006392  1.98551              0.006691      1.936807
N = 32  -2.486946 -2.499080    -2.486500    0.004879  1.989662     0.005059    1.950854 -2.490931 -2.486894          -2.486946 0.001602  1.996036             0.001624      1.981971
N = 64  -2.486946 -2.489985    -2.486917    0.001222  1.99738      0.001234    1.987073 -2.487943 -2.486943          -2.486946 0.000401  1.998875             0.000402      1.995088
N = 128 -2.486946 -2.487706    -2.486944    0.000306  1.999343     0.000306    1.996725 -2.487196 -2.486946          -2.486946 0.000100  1.999654             0.000100      1.998615

Convergence of FDM and FEM Temperature at x_bar = 0.500 and k_ratio = 0.5
          True T     FDM T  FDM Extrap T  FDM Exact  % Error  FDM Exact B  FDM Extrap  % Error  FDM Extrap B     FEM T  FEM Extrap T  FEM Exact % Error  FEM Exact B  FEM Extrap % Error  FEM Extrap B
N = 4    8.438605  9.674633           NaN   0.146473      NaN          NaN         NaN      NaN  8.652638          NaN          0.025364  NaN                      NaN           NaN
N = 8    8.438605  8.767916           NaN   0.039024  1.908190          NaN         NaN      NaN  8.432466          NaN          0.000727  5.123736                 NaN           NaN
N = 16   8.438605  8.522370       8.431179  0.009926  1.975035     0.010816    1.884659  8.429022  8.428967          0.001136  -0.642537            0.000006      5.998275
N = 32   8.438605  8.459639       8.438114  0.002493  1.993614     0.002551    1.968752  8.435145  8.431226          0.000410  1.469764             0.000465      -0.830074
N = 64   8.438605  8.443869       8.438574  0.000624  1.998394     0.000628    1.992015  8.437603  8.439252          0.000119  1.788181             0.000195      1.316727
N = 128  8.438605  8.439921       8.438603  0.000156  1.999598     0.000156    1.997993  8.438337  8.438650          0.000032  1.903464             0.000037      1.743720

Convergence of FDM and FEM Heat Convection with x_bar = 0.500 and k_ratio = 0.5
          True Q     FDM Q  FDM Extrap Q  FDM Exact  % Error FDM Exact B  FDM Extrap  % Error  FDM Extrap B     FEM Q  FEM Extrap Q  FEM Exact % Error FEM Exact B  FEM Extrap % Error  FEM Extrap B
N = 4   -3.529489 -4.044925           NaN   0.146037      [None]          NaN         NaN      NaN -3.695537          NaN          0.047046  [None]                 NaN           NaN
N = 8   -3.529489 -3.665089           NaN   0.038419  1.926443          NaN         NaN      NaN -3.572301          NaN          0.012130  1.955519               NaN           NaN
N = 16  -3.529489 -3.563869    -3.527097    0.009741  1.979723     0.010426    1.907890 -3.540313 -3.529099          -3.529456 0.003067  1.983847             0.003178      1.945806
N = 32  -3.529489 -3.538115    -3.529327    0.002444  1.994793     0.002490    1.974639 -3.532207 -3.529456          -3.529489 0.000770  1.993661             0.000779      1.980542
N = 64  -3.529489 -3.531648    -3.529479    0.000612  1.998689     0.000614    1.993491 -3.530170 -3.529486          -3.529489 0.000193  1.997282             0.000194      1.992449
N = 128 -3.529489 -3.530029    -3.529489    0.000153  1.999672     0.000153    1.998362 -3.529660 -3.529489          -3.529489 0.000048  1.998756             0.000048      1.996790

Convergence of FDM and FEM Temperature at x_bar = 0.500 and k_ratio = 1.0
          True T     FDM T  FDM Extrap T  FDM Exact  % Error  FDM Exact B  FDM Extrap  % Error  FDM Extrap B     FEM T  FEM Extrap T  FEM Exact % Error  FEM Exact B  FEM Extrap % Error  FEM Extrap B
N = 4    19.738549 20.447284           NaN   0.035906      NaN          NaN         NaN      NaN 19.679634          NaN          0.002985  NaN                      NaN           NaN
N = 8    19.738549 19.922720           NaN   0.009331  1.944198          NaN         NaN      NaN 19.641717          NaN          0.004906  -0.716842                NaN           NaN
N = 16   19.738549 19.785060    19.736081    0.002356  1.985391     0.002482    1.930009 19.703350 19.665192          19.665192 0.001783  1.459966             0.001940      -0.700878
N = 32   19.738549 19.750206    19.738390    0.000591  1.996303     0.000599    1.981722 19.728328 19.745349          19.738597 0.000518  1.784051             0.000862      1.303049
N = 64   19.738549 19.741465    19.738539    0.000148  1.999073     0.000148    1.995378 19.735813 19.739016          19.738597 0.000139  1.901696             0.000162      1.738548
N = 128  19.738549 19.739278    19.738548    0.000037  1.999768     0.000037    1.998841 19.737842 19.738597          19.738597 0.000036  1.952962             0.000038      1.883408

Convergence of FDM and FEM Heat Convection with x_bar = 0.500 and k_ratio = 1.0
          True Q     FDM Q  FDM Extrap Q  FDM Exact  % Error FDM Exact B  FDM Extrap  % Error  FDM Extrap B     FEM Q  FEM Extrap Q  FEM Exact % Error FEM Exact B  FEM Extrap % Error  FEM Extrap B
N = 4   -4.985127 -5.363661           NaN   0.075933      [None]          NaN         NaN      NaN -5.095071          NaN          0.022055  [None]                 NaN           NaN
N = 8   -4.985127 -5.082312           NaN   0.019495  1.961618          NaN         NaN      NaN -5.014775          NaN          0.005947  1.890731               NaN           NaN
N = 16  -4.985127 -5.009594    -4.984248    0.004908  1.989885     0.005085    1.951981 -4.992802 -4.984523          -4.984523 0.001540  1.949706             0.001661      1.869549
N = 32  -4.985127 -4.991254    -4.985069    0.001229  1.997436     0.001241    1.987353 -4.987077 -4.985061          -4.985061 0.000391  1.976014             0.000405      1.940629
N = 64  -4.985127 -4.986659    -4.985123    0.000307  1.999357     0.000308    1.996795 -4.985618 -4.985119          -4.985119 0.000098  1.988326             0.000100      1.971841
N = 128 -4.985127 -4.985510    -4.985126    0.000077  1.999839     0.000077    1.999196 -4.985250 -4.985126          -4.985126 0.000025  1.994248             0.000025      1.986336

Convergence of FDM and FEM Temperature at x_bar = 0.500 and k_ratio = 2.0
          True T     FDM T  FDM Extrap T  FDM Exact  % Error  FDM Exact B  FDM Extrap  % Error  FDM Extrap B     FEM T  FEM Extrap T  FEM Exact % Error  FEM Exact B  FEM Extrap % Error  FEM Extrap B
N = 4    36.441301 36.351342           NaN   0.002469      NaN          NaN         NaN      NaN 35.832852          NaN          0.016697  NaN                      NaN           NaN
N = 8    36.441301 36.412116           NaN   0.000801  1.624043          NaN         NaN      NaN 36.204196          NaN          0.006507  1.359606                 NaN           NaN
N = 16   36.441301 36.433533    36.445188    0.000213  1.909614     0.000320    1.504701 36.370470 36.505289          36.505289 0.001944  1.743070             0.003693      1.159187
N = 32   36.441301 36.439329    36.441479    0.000054  1.977577     0.000059    1.885733 36.422094 36.445339          36.441479 0.000527  1.882741             0.000638      1.687447
N = 64   36.441301 36.440806    36.441312    0.000014  1.994404     0.000014    1.971895 36.436309 36.441710          36.441479 0.000137  1.943854             0.000148      1.860649
N = 128  36.441301 36.441177    36.441302    0.000003  1.998602     0.000003    1.993004 36.440029 36.441348          36.441348 0.000035  1.972518             0.000036      1.933921

Convergence of FDM and FEM Heat Convection with x_bar = 0.500 and k_ratio = 2.0
          True Q     FDM Q  FDM Extrap Q  FDM Exact  % Error FDM Exact B  FDM Extrap  % Error  FDM Extrap B     FEM Q  FEM Extrap Q  FEM Exact % Error FEM Exact B  FEM Extrap % Error  FEM Extrap B
N = 4   -6.832006 -7.120389           NaN   0.042211      [None]          NaN         NaN      NaN -6.930078          NaN          0.014355  [None]                 NaN           NaN
N = 8   -6.832006 -6.905215           NaN   0.010716  1.977901          NaN         NaN      NaN -6.861111          NaN          0.004260  1.752561               NaN           NaN
N = 16  -6.832006 -6.850381    -6.831629    0.002690  1.994255     0.002745    1.972378 -6.839891 -6.830460          -6.831629 0.001154  1.88406              0.001381      1.700471
N = 32  -6.832006 -6.836605    -6.831982    0.000673  1.998549     0.000677    1.992819 -6.834055 -6.831842          -6.831982 0.000300  1.943973             0.000324      1.862416
N = 64  -6.832006 -6.833156    -6.832005    0.000168  1.999636     0.000169    1.998186 -6.832528 -6.831987          -6.831987 0.000076  1.972497             0.000079      1.934089
N = 128 -6.832006 -6.832294    -6.832006    0.000042  1.999909     0.000042    1.999545 -6.832138 -6.832004          -6.832004 0.000019  1.986381             0.000020      1.967780

Convergence of FDM and FEM Temperature at x_bar = 0.500 and k_ratio = 4.0
          True T     FDM T  FDM Extrap T  FDM Exact  % Error  FDM Exact B  FDM Extrap  % Error  FDM Extrap B     FEM T  FEM Extrap T  FEM Exact % Error  FEM Exact B  FEM Extrap % Error  FEM Extrap B
N = 4    55.383747 54.738450           NaN   0.011651      NaN          NaN         NaN      NaN 54.410234          NaN          0.017578  NaN                      NaN           NaN
N = 8    55.383747 55.211305           NaN   0.003114  1.903854          NaN         NaN      NaN 55.070107          NaN          0.005663  1.634093                 NaN           NaN
N = 16   55.383747 55.339880    55.387898    0.000792  1.974909     0.000867    1.878789 55.295589 55.412632          0.001592  1.830954             0.002112      1.549177
N = 32   55.383747 55.372732    55.384006    0.000199  1.993656     0.000204    1.968568 55.360424 55.386591          0.000421  1.918370             0.000472      1.798167
N = 64   55.383747 55.380090    55.383763    0.000050  1.998410     0.000050    1.992066 55.377752 55.384071          0.000108  1.959856             0.000114      1.903734
N = 128  55.383747 55.383057    55.383748    0.000012  1.999602     0.000012    1.998012 55.382227 55.383786          0.000027  1.980090             0.000028      1.952921

Convergence of FDM and FEM Heat Convection with x_bar = 0.500 and k_ratio = 4.0
          True Q     FDM Q  FDM Extrap Q  FDM Exact  % Error FDM Exact B  FDM Extrap  % Error  FDM Extrap B     FEM Q  FEM Extrap Q  FEM Exact % Error FEM Exact B  FEM Extrap % Error  FEM Extrap B
N = 4   -8.797049 -9.060288           NaN   0.029924      [None]          NaN         NaN      NaN -8.946633          NaN          0.017004  [None]                 NaN           NaN
N = 8   -8.797049 -8.864016           NaN   0.007612  1.974855          NaN         NaN      NaN -8.842519          NaN          0.005169  1.71798                NaN           NaN
N = 16  -8.797049 -8.813868    -8.796658    0.001912  1.993338     0.001956    1.968603 -8.809529 -8.794228          -8.796744 0.001419  1.865265             0.001740      1.658091
N = 32  -8.797049 -8.801259    -8.797023    0.000479  1.998309     0.000481    1.991674 -8.800315 -8.796744          -8.797023 0.000371  1.934016             0.000406      1.840087
N = 64  -8.797049 -8.798102    -8.797047    0.000120  1.999576     0.000120    1.997886 -8.797884 -8.797013          -8.797045 0.000095  1.967339             0.000099      1.922386
N = 128 -8.797049 -8.797312    -8.797049    0.000030  1.999894     0.000030    1.999469 -8.797260 -8.797045          -8.797045 0.000024  1.983751             0.000025      1.961743

Convergence of FDM and FEM Temperature at x_bar = 0.500 and k_ratio = 8.0
          True T     FDM T  FDM Extrap T  FDM Exact  % Error  FDM Exact B  FDM Extrap  % Error  FDM Extrap B     FEM T  FEM Extrap T  FEM Exact % Error  FEM Exact B  FEM Extrap % Error  FEM Extrap B
N = 4    72.105546 71.363029           NaN   0.010298      NaN          NaN         NaN      NaN 71.167084          NaN          0.013015  NaN                      NaN           NaN
N = 8    72.105546 71.911316           NaN   0.002694  1.934658          NaN         NaN      NaN 71.823222          NaN          0.003915  1.732945                 NaN           NaN
N = 16   72.105546 72.056411    72.108626    0.000681  1.982937     0.000724    1.917936 72.028173 72.121272          0.001073  1.867448             0.001291      1.678719
N = 32   72.105546 72.093226    72.105743    0.000171  1.995685     0.000174    1.978646 72.085293 72.107363          0.000281  1.933672             0.000306      1.843217
N = 64   72.105546 72.102464    72.105559    0.000043  1.998918     0.000043    1.994605 72.100365 72.105768          0.000072  1.966785             0.000075      1.922112
N = 128  72.105546 72.104776    72.105547    0.000011  1.999729     0.000011    1.998647 72.104236 72.105574          0.000018  1.983374             0.000019      1.961125

Convergence of FDM and FEM Heat Convection with x_bar = 0.500 and k_ratio = 8.0
          True Q     FDM Q  FDM Extrap Q  FDM Exact  % Error FDM Exact B  FDM Extrap  % Error  FDM Extrap B     FEM Q  FEM Extrap Q  FEM Exact % Error FEM Exact B  FEM Extrap % Error  FEM Extrap B
N = 4   -10.486214 -10.779857           NaN   0.028003      [None]          NaN         NaN      NaN -10.732095          NaN          0.023448  [None]                 NaN           NaN
N = 8   -10.486214 -10.561319           NaN   0.007162  1.967076          NaN         NaN      NaN -10.559111          NaN          0.006952  1.754021               NaN           NaN
N = 16  -10.486214 -10.505104    -10.485636    0.001801  1.991276     0.001857    1.958852 -10.506062    -10.482597          0.001893  1.876861             0.002238      1.705232
```

```
N = 32  -10.486214 -10.490944    -10.486176          0.000451     1.997786          0.000455     1.989096 -10.491393    -10.485787          0.000494     1.938138          0.000535     1.854587
N = 64  -10.486214 -10.487397    -10.486211          0.000113     1.999444          0.000113     1.997232 -10.487537    -10.486161          0.000126     1.968962          0.000131     1.927410
N = 128 -10.486214 -10.486510    -10.486214          0.000028     1.999861          0.000028     1.999305 -10.486548    -10.486207          0.000032     1.98445           0.000033     1.963687
```

Convergence of FDM and FEM Temperature at x_bar = 0.500 and k_ratio = 16.0

	True T	FDM T	FDM Extrap T	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM T	FEM Extrap T	FEM Exact % Error	FEM Exact B	FEM Extrap % Error	FEM Extrap B
N = 4	84.066993	83.499938	NaN		0.006745	NaN		NaN	NaN	83.389648	NaN	0.008057	NaN	NaN	NaN
N = 8	84.066993	83.920263	NaN		0.001745	1.950326		NaN	NaN	83.869497	NaN	0.002349	1.778069	NaN	NaN
N = 16	84.066993	84.029979	84.068733		0.000440	1.986999		0.000461	1.937740	84.013509	84.075262	0.000636	1.884628	0.000735	1.736399
N = 32	84.066993	84.057718	84.067105		0.000110	1.996710		0.000112	1.983738	84.053063	84.068040	0.000166	1.940876	0.000178	1.864285
N = 64	84.066993	84.064673	84.067000		0.000028	1.999175		0.000028	1.995887	84.063437	84.067126	0.000042	1.970029	0.000044	1.930747
N = 128	84.066993	84.066413	84.066994		0.000007	1.999794		0.000007	1.998969	84.066095	84.067010	0.000011	1.984906	0.000011	1.964966

Convergence of FDM and FEM Heat Convection with x_bar = 0.500 and k_ratio = 16.0

	True Q	FDM Q	FDM Extrap Q	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM Q	FEM Extrap Q	FEM Exact % Error	FEM Exact B	FEM Extrap % Error	FEM Extrap B
N = 4	-11.680625	-12.024326	NaN		0.029425	[None]		NaN	NaN	-12.024123	NaN	0.029407	[None]	NaN	NaN
N = 8	-11.680625	-11.768665	NaN		0.007537	1.964919		NaN	NaN	-11.780305	NaN	0.008534	1.784929	NaN	NaN
N = 16	-11.680625	-11.702777	-11.679901		0.001896	1.990732		0.001959	1.956136	-11.707559	-11.676625	0.002306	1.887865	0.002649	1.744873
N = 32	-11.680625	-11.686172	-11.680577		0.000475	1.997649		0.000479	1.988414	-11.687633	-11.680115	0.000600	1.942452	0.000644	1.868167
N = 64	-11.680625	-11.682012	-11.680622		0.000119	1.99941		0.000119	1.997062	-11.682413	-11.680560	0.000153	1.970807	0.000159	1.932611
N = 128	-11.680625	-11.680972	-11.680625		0.000030	1.999852		0.000030	1.999263	-11.681077	-11.680617	0.000039	1.985293	0.000039	1.965880

Convergence of FDM and FEM Temperature at x_bar = 0.637 and k_ratio = 0.0625

	True T	FDM T	FDM Extrap T	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM T	FEM Extrap T	FEM Exact % Error	FEM Exact B	FEM Extrap % Error	FEM Extrap B
N = 4	0.392138	0.964920	NaN		1.460667	NaN		NaN	NaN	0.149088	NaN	0.619806	NaN	NaN	NaN
N = 8	0.392138	0.534322	NaN		0.362587	2.010228		NaN	NaN	0.367363	NaN	0.063178	3.294328	NaN	NaN
N = 16	0.392138	0.427450	0.392169		0.090051	2.009513		0.089965	2.010465	0.386079	0.387834	0.015450	2.031788	0.004526	3.543820
N = 32	0.392138	0.400947	0.392207		0.022465	2.003074		0.022284	2.011646	0.390525	0.391911	0.004112	1.909892	0.003535	2.073563
N = 64	0.392138	0.394339	0.392144		0.005613	2.000819		0.005597	2.003825	0.391716	0.392151	0.001076	1.934512	0.001110	1.901067
N = 128	0.392138	0.392688	0.392138		0.001403	2.000209		0.001402	2.001022	0.392029	0.392142	0.000276	1.962964	0.000286	1.924564

Convergence of FDM and FEM Heat Convection with x_bar = 0.637 and k_ratio = 0.0625

	True Q	FDM Q	FDM Extrap Q	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM Q	FEM Extrap Q	FEM Exact % Error	FEM Exact B	FEM Extrap % Error	FEM Extrap B
N = 4	-1.241978	-1.892412	NaN		0.523708	[None]		NaN	NaN	-1.490265	NaN	0.199912	[None]	NaN	NaN
N = 8	-1.241978	-1.432463	NaN		0.153372	1.771728		NaN	NaN	-1.308439	NaN	0.053512	1.90142	NaN	NaN
N = 16	-1.241978	-1.292229	-1.230720		0.040461	1.922447		0.049978	1.713643	-1.258921	-1.240389	0.013642	1.971793	0.014941	1.876530
N = 32	-1.241978	-1.254729	-1.241041		0.010267	1.978528		0.011030	1.902870	-1.246235	-1.241866	0.003428	1.992665	0.003518	1.964720
N = 64	-1.241978	-1.245178	-1.241914		0.002577	1.99448		0.002628	1.973144	-1.243044	-1.241971	0.000858	1.998145	0.000864	1.990831
N = 128	-1.241978	-1.242779	-1.241974		0.000645	1.998612		0.000648	1.993098	-1.242245	-1.241978	0.000215	1.999539	0.000215	1.997680

Convergence of FDM and FEM Temperature at x_bar = 0.637 and k_ratio = 0.125

	True T	FDM T	FDM Extrap T	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM T	FEM Extrap T	FEM Exact % Error	FEM Exact B	FEM Extrap % Error	FEM Extrap B
N = 4	1.973771	2.884832	NaN		0.461584	NaN		NaN	NaN	1.950437	NaN	0.011822	NaN	NaN	NaN
N = 8	1.973771	2.211011	NaN		0.123206	1.905156		NaN	NaN	1.973971	NaN	0.010032	0.236949	NaN	NaN
N = 16	1.973771	2.035705	1.968139		0.031379	1.973570		0.034330	1.881035	1.965537	1.948881	0.004172	1.265896	0.008546	-1.710453
N = 32	1.973771	1.989328	1.973387		0.007882	1.993183		0.008078	1.966930	1.971213	1.976683	0.001296	1.686822	0.002767	1.026937
N = 64	1.973771	1.977665	1.973746		0.001973	1.998282		0.001985	1.991477	1.973063	1.973958	0.000358	1.854433	0.000453	1.617231
N = 128	1.973771	1.974744	1.973769		0.000493	1.999570		0.000494	1.997852	1.973585	1.973790	0.000094	1.929744	0.000104	1.826653

Convergence of FDM and FEM Heat Convection with x_bar = 0.637 and k_ratio = 0.125

	True Q	FDM Q	FDM Extrap Q	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM Q	FEM Extrap Q	FEM Exact % Error	FEM Exact B	FEM Extrap % Error	FEM Extrap B
N = 4	-1.758799	-2.268096	NaN		0.289571	[None]		NaN	NaN	-1.940183	NaN	0.103130	[None]	NaN	NaN
N = 8	-1.758799	-1.898303	NaN		0.079602	1.86304		NaN	NaN	-1.805860	NaN	0.026757	1.946449	NaN	NaN
N = 16	-1.758799	-1.794813	-1.754052		0.020477	1.958837		0.023238	1.828323	-1.770688	-1.758212	0.006760	1.984849	0.007096	1.933234
N = 32	-1.758799	-1.767870	-1.758449		0.005158	1.989132		0.005358	1.948491	-1.761780	-1.758758	0.001695	1.995926	0.001718	1.981123
N = 64	-1.758799	-1.761071	-1.758776		0.001292	1.997244		0.001305	1.986411	-1.759545	-1.758796	0.000424	1.998878	0.000426	1.994939
N = 128	-1.758799	-1.759367	-1.758797		0.000323	1.999308		0.000324	1.996554	-1.758985	-1.758799	0.000106	1.999671	0.000106	1.998614

Convergence of FDM and FEM Temperature at x_bar = 0.637 and k_ratio = 0.25

	True T	FDM T	FDM Extrap T	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM T	FEM Extrap T	FEM Exact % Error	FEM Exact B	FEM Extrap % Error	FEM Extrap B
N = 4	6.560619	7.452341	NaN		0.135920	NaN		NaN	NaN	6.572238	NaN	0.001771	NaN	NaN	NaN
N = 8	6.560619	6.799021	NaN		0.036338	1.903201		NaN	NaN	6.501798	NaN	0.008966	-2.339890	NaN	NaN
N = 16	6.560619	6.621332	6.554951		0.009254	1.973315		0.010127	1.878442	6.537168	6.525344	0.003575	1.326678	0.001812	0.993864
N = 32	6.560619	6.575870	6.560239		0.002325	1.993146		0.002383	1.966601	6.553589	6.567820	0.001072	1.738076	0.002167	1.106969
N = 64	6.560619	6.564437	6.560595		0.000582	1.998275		0.000586	1.991430	6.558712	6.561034	0.000291	1.881813	0.000354	1.680666
N = 128	6.560619	6.561574	6.560618		0.000146	1.999568		0.000146	1.997843	6.560124	6.560661	0.000076	1.943682	0.000082	1.859435

Convergence of FDM and FEM Heat Convection with x_bar = 0.637 and k_ratio = 0.25

	True Q	FDM Q	FDM Extrap Q	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM Q	FEM Extrap Q	FEM Exact % Error	FEM Exact B	FEM Extrap % Error	FEM Extrap B
N = 4	-2.501202	-2.888923	NaN		0.155014	[None]		NaN	NaN	-2.626242	NaN	0.049992	[None]	NaN	NaN
N = 8	-2.501202	-2.603336	NaN		0.040834	1.924555		NaN	NaN	-2.533474	NaN	0.012903	1.954046	NaN	NaN
N = 16	-2.501202	-2.527108	-2.499354		0.010358	1.979071		0.011105	1.905549	-2.509362	-2.500895	0.003263	1.983522	0.003386	1.943932
N = 32	-2.501202	-2.507703	-2.501075		0.002599	1.994616		0.002650	1.973826	-2.503251	-2.501176	0.000819	1.99364	0.000830	1.980114
N = 64	-2.501202	-2.502828	-2.501194		0.000650	1.998644		0.000654	1.993269	-2.501715	-2.501199	0.000205	1.997315	0.000206	1.992409
N = 128	-2.501202	-2.501608	-2.501201		0.000163	1.99966		0.000163	1.998305	-2.501330	-2.501201	0.000051	1.998785	0.000051	1.996825

Convergence of FDM and FEM Temperature at x_bar = 0.637 and k_ratio = 0.5

	True T	FDM T	FDM Extrap T	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM T	FEM Extrap T	FEM Exact % Error	FEM Exact B	FEM Extrap % Error	FEM Extrap B
N = 4	16.08921	16.377482	NaN		0.017917	NaN		NaN	NaN	15.648397	NaN	0.027398	NaN	NaN	NaN
N = 8	16.08921	16.157318	NaN		0.004233	2.081543		NaN	NaN	15.873740	NaN	0.013392	1.032680	NaN	NaN
N = 16	16.08921	16.105900	16.090233		0.001037	2.028836		0.000974	2.098246	16.020446	16.294141	0.004274	1.647759	0.016797	0.619196
N = 32	16.08921	16.093360	16.089315		0.000258	2.007904		0.000251	2.035696	16.070057	16.095406	0.001190	1.844072	0.001575	1.564197
N = 64	16.08921	16.090246	16.089218		0.000064	2.002023		0.000064	2.009856	16.084171	16.089784	0.000313	1.926462	0.000349	1.813482
N = 128	16.08921	16.089469	16.089211		0.000016	2.000509		0.000016	2.002527	16.087919	16.089273	0.000080	1.964289	0.000084	1.913195

Convergence of FDM and FEM Heat Convection with x_bar = 0.637 and k_ratio = 0.5

	True Q	FDM Q	FDM Extrap Q	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM Q	FEM Extrap Q	FEM Exact % Error	FEM Exact B	FEM Extrap % Error	FEM Extrap B
N = 4	-3.564213	-3.858649	NaN		0.082609	[None]		NaN	NaN	-3.651412	NaN	0.024465	[None]	NaN	NaN
N = 8	-3.564213	-3.639854	NaN		0.021222	1.960716		NaN	NaN	-3.587897	NaN	0.006645	1.88039	NaN	NaN
N = 16	-3.564213	-3.583260	-3.563514		0.005344	1.989586		0.005541	1.950868	-3.570374	-3.563697	0.001728	1.942759	0.001874	1.857806
N = 32	-3.564213	-3.568983	-3.564167		0.001338	1.997356		0.001351	1.986980	-3.565733	-3.564153	0.000441	1.972281	0.000457	1.932222
N = 64	-3.564213	-3.565406	-3.564210		0.000335	1.999336		0.000336	1.996695	-3.564609	-3.564206	0.000111	1.986436	0.000113	1.967471
N = 128	-3.564213	-3.564511	-3.564213		0.000084	1.999834		0.000084	1.999170	-3.564312	-3.564212	0.000028	1.993303	0.000028	1.984125

Convergence of FDM and FEM Temperature at x_bar = 0.637 and k_ratio = 1.0

	True T	FDM T	FDM Extrap T	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM T	FEM Extrap T	FEM Exact % Error	FEM Exact B	FEM Extrap % Error	FEM Extrap B
N = 4	31.182398	30.514477	NaN		0.021420	NaN		NaN	NaN	29.932292	NaN	0.040090	NaN	NaN	NaN
N = 8	31.182398	30.989359	NaN		0.006191	1.790787		NaN	NaN	30.733838	NaN	0.014385	1.478678	NaN	NaN
N = 16	31.182398	31.132234	31.193717		0.001609	1.944144		0.001971	1.732826	31.050756	31.258002	0.004222	1.768681	0.006630	1.338677
N = 32	31.182398	31.169733	31.183078		0.000406	1.985785		0.000428	1.928804	31.146901	31.188771	0.001138	1.890838	0.001342	1.720830
N = 64	31.182398	31.179224	31.182440		0.000102	1.996430		0.000103	1.982208	31.173192	31.183087	0.000295	1.947032	0.000317	1.870631
N = 128	31.182398	31.181604	31.182401		0.000025	1.999106		0.000026	1.995536	31.180055	31.182479	0.000075	1.973922	0.000078	1.937733

Convergence of FDM and FEM Heat Convection with x_bar = 0.637 and k_ratio = 1.0

	True Q	FDM Q	FDM Extrap Q	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM Q	FEM Extrap Q	FEM Exact % Error	FEM Exact B	FEM Extrap % Error	FEM Extrap B
N = 4	-4.985127	-5.229953	NaN		0.049111	[None]		NaN	NaN	-5.078401	NaN	0.018711	[None]	NaN	NaN
N = 8	-4.985127	-5.047653	NaN		0.012543	1.969225		NaN	NaN	-5.013420	NaN	0.005676	1.720991	NaN	NaN
N = 16	-4.985127	-5.000849	-4.984681		0.003154	1.991676		0.003243	1.961604	-4.992897	-4.983423	0.001559	1.864391	0.001901	1.662746
N = 32	-4.985127	-4.989063	-4.985096		0.000790	1.997874		0.000796	1.989601	-4.987161	-4.984935	0.000408	1.933604	0.000446	1.839028
N = 64	-4.985127	-4.986111	-4.985125		0.000197	1.999466		0.000198	1.997343	-4.985647	-4.985104	0.000104	1.967248	0.000109	1.921860
N = 128	-4.985127	-4.985373	-4.985126		0.000049	1.999866		0.000049	1.999332	-4.985258	-4.985124	0.000026	1.983749	0.000027	1.961621

Convergence of FDM and FEM Temperature at x_bar = 0.637 and k_ratio = 2.0

	True T	FDM T	FDM Extrap T	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM T	FEM Extrap T	FEM Exact % Error	FEM Exact B	FEM Extrap % Error	FEM Extrap B
N = 4	49.706164	48.318359	NaN		0.027920	NaN		NaN	NaN	47.861343	NaN	0.037115	NaN	NaN	NaN
N = 8	49.706164	49.325368	NaN		0.007661	1.865716		NaN	NaN	49.108745	NaN	0.012019	1.626666	NaN	NaN
N = 16	49.706164	49.608556	49.719351		0.001964	1.963950		0.002228	1.830245	49.536981	49.760850	0.003404	1.820156	0.004499	1.542449
N = 32	49.706164	49.681606	49.707000		0.000494	1.990813		0.000511	1.954806	49.661198	49.711952	0.000905	1.911699	0.001021	1.785533
N = 64	49.706164	49.700015	49.706216		0.000124	1.997692		0.000125	1.988508	49.694576	49.706841	0.000233	1.956261	0.000247	1.895902
N = 128	49.706164	49.704626	49.706167		0.000031	1.999422		0.000031	1.997114	49.703223	49.706246	0.000059	1.978235	0.000061	1.948711

Convergence of FDM and FEM Heat Convection with x_bar = 0.637 and k_ratio = 2.0

	True Q	FDM Q	FDM Extrap Q	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM Q	FEM Extrap Q	FEM Exact % Error	FEM Exact B	FEM Extrap % Error	FEM Extrap B
N = 4	-6.610807	-6.872679	NaN		0.039613	[None]		NaN	NaN	-6.783079	NaN	0.026059	[None]	NaN	NaN
N = 8	-6.610807	-6.678556	NaN		0.010248	1.950595		NaN	NaN	-6.664558	NaN	0.008131	1.680312	NaN	NaN
N = 16	-6.610807	-6.627905	-6.610024		0.002586	1.986349		0.002705	1.938323	-6.625791	-6.606947	0.002267	1.842832	0.002852	1.612237
N = 32	-6.610807	-6.615092	-6.610753		0.000648	1.996494		0.000656	1.982940	-6.614761	-6.610374	0.000598	1.922142	0.000664	1.813308
N = 64	-6.610807	-6.611879	-6.610804		0.000162	1.999117		0.000163	1.995618	-6.611822	-6.610755	0.000154	1.961274	0.000161	1.908370
N = 128	-6.610807	-6.611075	-6.610807		0.000041	1.999779		0.000041	1.998897	-6.611064	-6.610801	0.000039	1.980691	0.000040	1.954625

Convergence of FDM and FEM Temperature at x_bar = 0.637 and k_ratio = 4.0

	True T	FDM T	FDM Extrap T	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM T	FEM Extrap T	FEM Exact % Error	FEM Exact B	FEM Extrap % Error	FEM Extrap B
N = 4	67.384857	65.895105	NaN		0.022108	NaN		NaN	NaN	65.562359	NaN	0.027046	NaN	NaN	NaN
N = 8	67.384857	66.985431	NaN		0.005928	1.899072		NaN	NaN	66.822621	NaN	0.008344	1.696669	NaN	NaN
N = 16	67.384857	67.283107	67.394899		0.001510	1.972907		0.001659	1.872943	67.228445	67.421196	0.002321	1.845831	0.002859	1.634795
N = 32	67.384857	67.359297	67.385506		0.000379	1.993096		0.000389	1.966071	67.343581	67.389183	0.000613	1.921976	0.000677	1.817524
N = 64	67.384857	67.378459	67.384898		0.000095	1.998265		0.000096	1.991365	67.374253	67.385391	0.000157	1.960712	0.000165	1.908339
N = 128	67.384857	67.383257	67.384859		0.000024	1.999566		0.000024	1.997832	67.382169	67.384923	0.000040	1.980281	0.000041	1.954008

Convergence of FDM and FEM Heat Convection with x_bar = 0.637 and k_ratio = 4.0

	True Q	FDM Q	FDM Extrap Q	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM Q	FEM Extrap Q	FEM Exact % Error	FEM Exact B	FEM Extrap % Error	FEM Extrap B
N = 4	-8.118713	-8.460794	NaN		0.042135	[None]		NaN	NaN	-8.433519	NaN	0.038775	[None]	NaN	NaN
N = 8	-8.118713	-8.208090	NaN		0.011009	1.936363		NaN	NaN	-8.214918	NaN	0.011850	1.710279	NaN	NaN
N = 16	-8.118713	-8.141329	-8.117358		0.002786	1.982599		0.002953	1.920359	-8.145364	-8.112905	0.003283	1.851968	0.004001	1.652077
N = 32	-8.118713	-8.124385	-8.118622		0.000699	1.995544		0.000710	1.978241	-8.125732	-8.118012	0.000864	1.924895	0.000951	1.824974
N = 64	-8.118713	-8.120132	-8.118708		0.000175	1.998879		0.000175	1.994429	-8.120515	-8.118626	0.000222	1.962136	0.000233	1.911811
N = 128	-8.118713	-8.119068	-8.118713		0.000044	1.999719		0.000044	1.998599	-8.119170	-8.118703	0.000056	1.980985	0.000058	1.955685

Convergence of FDM and FEM Temperature at x_bar = 0.637 and k_ratio = 8.0

	True T	FDM T	FDM Extrap T	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM T	FEM Extrap T	FEM Exact	% Error	FEM Exact B	FEM Extrap	% Error	FEM Extrap B
N = 4	80.863285	79.714831	NaN	0.014202	NaN		NaN	NaN	NaN	79.498914	NaN	0.016873	NaN		NaN	NaN	NaN
N = 8	80.863285	80.559732	NaN	0.003754	1.919673		NaN	NaN	NaN	80.453254	NaN	0.005071	1.734431		NaN	NaN	NaN
N = 16	80.863285	80.786251	80.869227	0.000953	1.978385		0.001026	1.899150		80.750296	80.884534	0.001397	1.859558		0.001660	1.683834	
N = 32	80.863285	80.843953	80.863675	0.000239	1.994488		0.000244	1.972949		80.833578	80.866025	0.000367	1.927308		0.000401	1.834600	
N = 64	80.863285	80.858447	80.863310	0.000060	1.998615		0.000060	1.993108		80.855665	80.863637	0.000094	1.962954		0.000099	1.914803	
N = 128	80.863285	80.862075	80.863286	0.000015	1.999653		0.000015	1.998268		80.861355	80.863330	0.000024	1.981291		0.000024	1.956682	

Convergence of FDM and FEM Heat Convection with x_bar = 0.637 and k_ratio = 8.0

	True Q	FDM Q	FDM Extrap Q	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM Q	FEM Extrap Q	FEM Exact	% Error	FEM Exact B	FEM Extrap	% Error	FEM Extrap B
N = 4	-9.254592	-9.698320	NaN	0.047947	[None]		NaN	NaN	NaN	-9.722544	NaN	0.050564	[None]		NaN	NaN	NaN
N = 8	-9.254592	-9.370579	NaN	0.012533	1.935704		NaN	NaN	NaN	-9.394792	NaN	0.015149	1.738869		NaN	NaN	NaN
N = 16	-9.254592	-9.283942	-9.252810	0.003171	1.982521		0.003365	1.919493		-9.293169	-9.247500	0.004169	1.86165		0.004939	1.689379	
N = 32	-9.254592	-9.261952	-9.254472	0.000795	1.995531		0.000808	1.978140		-9.264727	-9.253673	0.001095	1.928326		0.001195	1.837124	
N = 64	-9.254592	-9.256433	-9.254584	0.000199	1.998876		0.000200	1.994413		-9.257191	-9.254473	0.000281	1.963456		0.000294	1.916010	
N = 128	-9.254592	-9.255052	-9.254591	0.000050	1.999719		0.000050	1.998595		-9.255250	-9.254577	0.000071	1.98154		0.000073	1.957273	

Convergence of FDM and FEM Temperature at x_bar = 0.637 and k_ratio = 16.0

	True T	FDM T	FDM Extrap T	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM T	FEM Extrap T	FEM Exact	% Error	FEM Exact B	FEM Extrap	% Error	FEM Extrap B
N = 4	89.525116	88.797458	NaN	0.008128	NaN		NaN	NaN	NaN	88.671094	NaN	0.009539	NaN		NaN	NaN	NaN
N = 8	89.525116	89.334419	NaN	0.002130	1.931977		NaN	NaN	NaN	89.272101	NaN	0.002826	1.755051		NaN	NaN	NaN
N = 16	89.525116	89.476830	89.528233	0.000539	1.981624		0.000574	1.914748		89.455749	89.536559	0.000775	1.866912		0.000903	1.710436	
N = 32	89.525116	89.513005	89.525323	0.000135	1.995309		0.000138	1.977014		89.506913	89.526672	0.000203	1.930093		0.000221	1.843749	
N = 64	89.525116	89.522085	89.525129	0.000034	1.998821		0.000034	1.994135		89.520450	89.525321	0.000052	1.964098		0.000054	1.918185	
N = 128	89.525116	89.524358	89.525116	0.000008	1.999705		0.000008	1.998526		89.523934	89.525142	0.000013	1.981797		0.000013	1.958049	

Convergence of FDM and FEM Heat Convection with x_bar = 0.637 and k_ratio = 16.0

	True Q	FDM Q	FDM Extrap Q	FDM Exact	% Error	FDM Exact B	FDM Extrap	% Error	FDM Extrap B	FEM Q	FEM Extrap Q	FEM Exact	% Error	FEM Exact B	FEM Extrap	% Error	FEM Extrap B
N = 4	-9.980639	-10.508396	NaN	0.052878	[None]		NaN	NaN	NaN	-10.567911	NaN	0.058841	[None]		NaN	NaN	NaN
N = 8	-9.980639	-10.118258	NaN	0.013789	1.939192		NaN	NaN	NaN	-10.154415	NaN	0.017411	1.756795		NaN	NaN	NaN
N = 16	-9.980639	-10.015440	-9.978646	0.003487	1.983481		0.003687	1.923889		-10.028253	-9.972858	0.004771	1.867755		0.005555	1.712595	
N = 32	-9.980639	-9.989364	-9.980505	0.000874	1.995777		0.000888	1.979342		-9.993130	-9.979579	0.001252	1.930509		0.001358	1.844764	
N = 64	-9.980639	-9.982822	-9.980630	0.000219	1.998938		0.000220	1.994720		-9.983840	-9.980499	0.000321	1.964305		0.000335	1.918678	
N = 128	-9.980639	-9.981185	-9.980638	0.000055	1.999734		0.000055	1.998673		-9.981449	-9.980621	0.000081	1.9819		0.000083	1.958292	

5 Python Code

5.1 main.py

```
# main.py

import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import math

import case1
import case2
import common
import Bar


bar1 = Bar.Bar(
    radius=0.1,
    k=0.5,
    h=0.25
)
bar2 = Bar.Bar(
    radius=0.1,
    k=2,
    h=0.25
)

data_dict = {}
data_dict["k_ratios"] = np.array([1/16, 1/8, 1/4, 1/2, 1, 2, 4, 8, 16])
data_dict["x_bar_values"] = np.array([1/2, 2/np.pi])
data_dict["length"] = 1

def section_1():
    '''Analytical solutions with varying k_ratio'''
    print("Section 1: Analytical")

    l = data_dict["length"]

    # Calculate the results for every k ratio for x_bar_1
    x_bar_1 = data_dict["x_bar_values"][0]

    data_dict["section1"] = {}
    data_dict["section1"]["x_bar_1"] = {}
    data_dict["section1"]["x_bar_1"]["x_values_fine"] = np.linspace(0, l, 50)
    data_dict["section1"]["x_bar_1"]["x_values_fine"] =
        np.sort(np.append(data_dict["section1"]["x_bar_1"]["x_values_fine"], x_bar_1))
    data_dict["section1"]["x_bar_1"]["x_values_course"] = np.linspace(0, l, 9)
    data_dict["section1"]["x_bar_1"]["x_values_course"] =
        np.sort(np.append(data_dict["section1"]["x_bar_1"]["x_values_course"], x_bar_1))
```

```
results = []
results_course = []
for k_ratio in data_dict["k_ratios"]:
    Bar.set_k_ratio(k_ratio, bar1, bar2) # Set the k value of bar2 = k*bar1
    result =
        case1.get_analytical_datapoints(data_dict["section1"]["x_bar_1"]["x_values_fine"],
                                        bar1=bar1,
                                        bar2=bar2,
                                        x_bar=x_bar_1,
                                        l=l)
    result_course =
        case1.get_analytical_datapoints(data_dict["section1"]["x_bar_1"]["x_values_course"],
                                        bar1=bar1,
                                        bar2=bar2,
                                        x_bar=x_bar_1,
                                        l=l)
    results.append(result)
    results_course.append(result_course)

data_dict["section1"]["x_bar_1"]["temp_results"] = np.array(results)

df_x_bar_1 = pd.DataFrame(results_course, index=[f'k2/k1 = {a}' for a in
    data_dict["k_ratios"]], columns=[f'x = {x:.3f}' for x in
    data_dict["section1"]["x_bar_1"]["x_values_course"]])

print("Analytical Temperature Results x_bar = 0.5:")
print(df_x_bar_1.to_string())
print("\n" * 2)

# Plotting x_bar_1
plt.figure(figsize=(10, 6))
for idx, k_ratio in enumerate(data_dict["k_ratios"]):
    plt.plot(data_dict["section1"]["x_bar_1"]["x_values_fine"],
        data_dict["section1"]["x_bar_1"]["temp_results"][idx], label=f'k2/k1 =
        {k_ratio}')
plt.axvline(x=x_bar_1, color='r', linestyle='--', label='x_bar')
plt.xlabel('Position (cm)')
plt.ylabel('Temperature (C)')
plt.legend()
plt.grid(True)
plt.savefig('images/section1/x_bar_1/analytical_temperature_vs_position.png', dpi=300)
#plt.show()
plt.clf()

# Calculate the temperature results for every k ratio for x_bar_1
x_bar_2 = data_dict["x_bar_values"][1]

data_dict["section1"]["x_bar_2"] = {}
data_dict["section1"]["x_bar_2"]["x_values_fine"] = np.linspace(0, l, 50)
```

```
data_dict["section1"]["x_bar_2"]["x_values_fine"] =
    np.sort(np.append(data_dict["section1"]["x_bar_2"]["x_values_fine"], x_bar_2))
data_dict["section1"]["x_bar_2"]["x_values_course"] = np.linspace(0, 1, 9)
data_dict["section1"]["x_bar_2"]["x_values_course"] =
    np.sort(np.append(data_dict["section1"]["x_bar_2"]["x_values_course"], x_bar_2))

results = []
results_course = []
for k_ratio in data_dict["k_ratios"]:
    Bar.set_k_ratio(k_ratio, bar1, bar2)
    result =
        case1.get_analytical_datapoints(data_dict["section1"]["x_bar_2"]["x_values_fine"],
                                        bar1=bar1,
                                        bar2=bar2,
                                        x_bar=x_bar_2,
                                        l=l)
    result_course =
        case1.get_analytical_datapoints(data_dict["section1"]["x_bar_2"]["x_values_course"],
                                        bar1=bar1,
                                        bar2=bar2,
                                        x_bar=x_bar_2,
                                        l=l)
    results.append(result)
    results_course.append(result_course)

data_dict["section1"]["x_bar_2"]["temp_results"] = np.array(results)

df_x_bar_2 = pd.DataFrame(results_course, index=[f'k2/k1 = {a}' for a in
    data_dict["k_ratios"]], columns=[f'x = {x:.3f}' for x in
    data_dict["section1"]["x_bar_2"]["x_values_course"]])

print("Analytical Temperature Results x_bar = 2/pi:")
print(df_x_bar_2.to_string())
print("\n" * 2)

# Plotting x_bar_2
plt.figure(figsize=(10, 6))
for idx, k_ratio in enumerate(data_dict["k_ratios"]):
    plt.plot(data_dict["section1"]["x_bar_2"]["x_values_fine"],
        data_dict["section1"]["x_bar_2"]["temp_results"][idx], label=f'k2/k1 =
        {k_ratio}')
plt.axvline(x=x_bar_2, color='r', linestyle='--', label='x_bar')
plt.xlabel('Position (cm)')
plt.ylabel('Temperature (C)')
plt.legend()
plt.grid(True)
plt.savefig('images/section1/x_bar_2/analytical_temperature_vs_position.png', dpi=300)
#plt.show()
plt.clf()
```

```
# Calculate the analytical heat convection leaving the rod at x=l for varying k ratio

results_1 = []
results_2 = []
for k_ratio in data_dict["k_ratios"]:
    Bar.set_k_ratio(k_ratio, bar1, bar2)
    result_1 = case1.get_analytical_heat_convection(x=l,
                                                    bar1=bar1,
                                                    bar2=bar2,
                                                    x_bar=x_bar_1,
                                                    l=l)
    result_2 = case1.get_analytical_heat_convection(x=l,
                                                    bar1=bar1,
                                                    bar2=bar2,
                                                    x_bar=x_bar_2,
                                                    l=l)
    results_1.append([result_1])
    results_2.append([result_2])

data_dict["section1"]["x_bar_1"]["heat_results"] = np.array(results_1)
data_dict["section1"]["x_bar_2"]["heat_results"] = np.array(results_2)

df_x_bar_1 = pd.DataFrame(results_1, index=[f'k2/k1 = {a}' for a in
    data_dict["k_ratios"]], columns=["Heat Convection"])
df_x_bar_2 = pd.DataFrame(results_2, index=[f'k2/k1 = {a}' for a in
    data_dict["k_ratios"]], columns=["Heat Convection"])

print("Analytical Heat Convection Results x_bar = 0.5:")
print(df_x_bar_1.to_string())
print("\n" * 2)

print("Analytical Heat Convection Results x_bar = 2/pi:")
print(df_x_bar_2.to_string())
print("\n" * 2)

# Plotting x_bar_1 heat convection
plt.figure(figsize=(10, 6))
plt.plot(data_dict["k_ratios"], data_dict["section1"]["x_bar_1"]["heat_results"],
    label=f'Heat Convection')
plt.xlabel('k2/k1')
plt.ylabel('Q_Dot (W)')
plt.legend()
plt.grid(True)
plt.savefig('images/section1/x_bar_1/analytical_heat_convection_vs_k_ratio.png',
    dpi=300)
#plt.show()
plt.clf()
```

```
    # Plotting x_bar_2 heat convection
    plt.figure(figsize=(10, 6))
    plt.plot(data_dict["k_ratios"], data_dict["section1"]["x_bar_2"]["heat_results"],
        label=f'Heat Convection')
    plt.xlabel('k2/k1')
    plt.ylabel('Q_Dot (W)')
    plt.legend()
    plt.grid(True)
    plt.savefig('images/section1/x_bar_2/analytical_heat_convection_vs_k_ratio.png',
        dpi=300)
    #plt.show()
    plt.clf()


def section_2():
    '''FDM and FEM with varying k_ratio'''
    print("Section 2: FDM and FEM with Varying k_ratio")

    l = data_dict["length"]

    data_dict["section2"] = {}
    data_dict["section2"]["num_sections"] = 8


    # Calculate the results for every k ratio for x_bar_1
    x_bar_1 = data_dict["x_bar_values"][0]

    # For every k_ratio, calculate the FDM and FEM temperature results
    fdm_results = []
    fem_results = []
    x_fdm = []
    x_fem = []
    for k_ratio in data_dict["k_ratios"]:
        Bar.set_k_ratio(k_ratio, bar1, bar2) # Set the k value of bar2 = k*bar1
        fdm_temps, x_fdm = case1.solve(bar1=bar1,
                                       bar2=bar2,
                                       num_sections=data_dict["section2"]["num_sections"],
                                       x_bar=x_bar_1,
                                       l=l,
                                       method="FDM")
        fem_temps, x_fem = case1.solve(bar1=bar1,
                                       bar2=bar2,
                                       num_sections=data_dict["section2"]["num_sections"],
                                       x_bar=x_bar_1,
                                       l=l,
                                       method="FEM")

        fdm_results.append(fdm_temps)
        fem_results.append(fem_temps)
```

```
data_dict["section2"]["x_bar_1"] = {}
data_dict["section2"]["x_bar_1"]["x_values"] = x_fdm # fdm and fem use same x_values
data_dict["section2"]["x_bar_1"]["fdm_results"] = np.array(fdm_results)
data_dict["section2"]["x_bar_1"]["fem_results"] = np.array(fem_results)

df_fdm = pd.DataFrame(fdm_results, index=[f'k2/k1 = {a}' for a in
    data_dict["k_ratios"]], columns=[f'x = {x:.3f}' for x in x_fdm])
df_fem = pd.DataFrame(fem_results, index=[f'k2/k1 = {a}' for a in
    data_dict["k_ratios"]], columns=[f'x = {x:.3f}' for x in x_fem])

print("FDM Temperature Results x_bar = 0.5:")
print(df_fdm.to_string())
print("\n" * 2)
print("FEM Temperature Results x_bar = 0.5:")
print(df_fem.to_string())
print("\n" * 2)

# Now that the data is gathered for FDM and FEM, plot it
# Plotting x_bar_1, FDM
plt.figure(figsize=(10, 6))
for idx, k_ratio in enumerate(data_dict["k_ratios"]):
    plt.plot(data_dict["section2"]["x_bar_1"]["x_values"],
        data_dict["section2"]["x_bar_1"]["fdm_results"][idx], label=f'k2/k1 =
        {k_ratio}')
plt.axvline(x=x_bar_1, color='r', linestyle='--', label='x_bar')
plt.xlabel('Position (cm)')
plt.ylabel('Temperature (C)')
plt.legend()
plt.grid(True)
plt.savefig('images/section2/x_bar_1/fdm_temperature_vs_position.png', dpi=300)
#plt.show()
plt.clf()

# Plotting x_bar_1, FEM
plt.figure(figsize=(10, 6))
for idx, k_ratio in enumerate(data_dict["k_ratios"]):
    plt.plot(data_dict["section2"]["x_bar_1"]["x_values"],
        data_dict["section2"]["x_bar_1"]["fem_results"][idx], label=f'k2/k1 =
        {k_ratio}')
plt.axvline(x=x_bar_1, color='r', linestyle='--', label='x_bar')
plt.xlabel('Position (cm)')
plt.ylabel('Temperature (C)')
plt.legend()
plt.grid(True)
plt.savefig('images/section2/x_bar_1/fem_temperature_vs_position.png', dpi=300)
#plt.show()
plt.clf()


# Calculate the results for every k ratio for x_bar_2
```

```
x_bar_2 = data_dict["x_bar_values"][1]

# For every k_ratio, calculate the FDM and FEM temperature results
fdm_results = []
fem_results = []
x_fdm = []
x_fem = []
for k_ratio in data_dict["k_ratios"]:
    Bar.set_k_ratio(k_ratio, bar1, bar2) # Set the k value of bar2 = k*bar1
    fdm_temps, x_fdm = case1.solve(bar1=bar1,
                                   bar2=bar2,
                                   num_sections=data_dict["section2"]["num_sections"],
                                   x_bar=x_bar_2,
                                   l=l,
                                   method="FDM")
    fem_temps, x_fem = case1.solve(bar1=bar1,
                                   bar2=bar2,
                                   num_sections=data_dict["section2"]["num_sections"],
                                   x_bar=x_bar_2,
                                   l=l,
                                   method="FEM")

    fdm_results.append(fdm_temps)
    fem_results.append(fem_temps)

data_dict["section2"]["x_bar_2"] = {}
data_dict["section2"]["x_bar_2"]["x_values"] = x_fdm # fdm and fem use same x_values
data_dict["section2"]["x_bar_2"]["fdm_results"] = np.array(fdm_results)
data_dict["section2"]["x_bar_2"]["fem_results"] = np.array(fem_results)

df_fdm = pd.DataFrame(fdm_results, index=[f'k2/k1 = {a}' for a in
    data_dict["k_ratios"]], columns=[f'x = {x:.3f}' for x in x_fdm])
df_fem = pd.DataFrame(fem_results, index=[f'k2/k1 = {a}' for a in
    data_dict["k_ratios"]], columns=[f'x = {x:.3f}' for x in x_fem])

print("FDM Temperature Results x_bar = 2/pi:")
print(df_fdm.to_string())
print("\n" * 2)
print("FEM Temperature Results x_bar = 2/pi:")
print(df_fem.to_string())
print("\n" * 2)

# Plotting x_bar_2, FDM
plt.figure(figsize=(10, 6))
for idx, k_ratio in enumerate(data_dict["k_ratios"]):
    plt.plot(data_dict["section2"]["x_bar_2"]["x_values"],
        data_dict["section2"]["x_bar_2"]["fdm_results"][idx], label=f'k2/k1 =
        {k_ratio}')
plt.axvline(x=x_bar_2, color='r', linestyle='--', label='x_bar')
plt.xlabel('Position (cm)')
```

```
plt.ylabel('Temperature (C)')
plt.legend()
plt.grid(True)
plt.savefig('images/section2/x_bar_2/fdm_temperature_vs_position.png', dpi=300)
#plt.show()
plt.clf()

# Plotting x_bar_2, FEM
plt.figure(figsize=(10, 6))
for idx, k_ratio in enumerate(data_dict["k_ratios"]):
    plt.plot(data_dict["section2"]["x_bar_2"]["x_values"],
        data_dict["section2"]["x_bar_2"]["fem_results"][idx], label=f'k2/k1 =
        {k_ratio}')
plt.axvline(x=x_bar_2, color='r', linestyle='--', label='x_bar')
plt.xlabel('Position (cm)')
plt.ylabel('Temperature (C)')
plt.legend()
plt.grid(True)
plt.savefig('images/section2/x_bar_2/fem_temperature_vs_position.png', dpi=300)
#plt.show()
plt.clf()



# After calculating temperature values, extract the heat convection at x=l

# x_bar_1
# for every k ratio, calculate the heat convection from fdm and fem temp results
fdm_heat_results = []
fem_heat_results = []
for i, k_ratio in enumerate(data_dict["k_ratios"]):
    Bar.set_k_ratio(k_ratio, bar1, bar2) # Set the k value of bar2 = k*bar1
    dx = data_dict["section2"]["x_bar_1"]["x_values"][-1] -
        data_dict["section2"]["x_bar_1"]["x_values"][-2]
    (fdm_t0, fdm_t1) = data_dict["section2"]["x_bar_1"]["fdm_results"][i][-2:]
    fdm_heat_result = common.fdm_heat_extraction(fdm_t0, fdm_t1, dx, bar2, order=2)
    fdm_heat_results.append(fdm_heat_result)

    (fem_t0, fem_t1) = data_dict["section2"]["x_bar_1"]["fem_results"][i][-2:]
    fem_heat_result = common.fem_heat_extraction(fem_t0, fem_t1, dx, bar2)
    fem_heat_results.append(fem_heat_result)


data_dict["section2"]["x_bar_1"]["fdm_heat_results"] = np.array(fdm_heat_results)
data_dict["section2"]["x_bar_1"]["fem_heat_results"] = np.array(fem_heat_results)

df_fdm = pd.DataFrame(fdm_heat_results, index=[f'k2/k1 = {a}' for a in
    data_dict["k_ratios"]], columns=["Heat Convection"])
df_fem = pd.DataFrame(fem_heat_results, index=[f'k2/k1 = {a}' for a in
    data_dict["k_ratios"]], columns=["Heat Convection"])
```

```
print("FDM Heat Convection Results x_bar = 0.5:")
print(df_fdm.to_string())
print("\n" * 2)

print("FEM Heat Convection Results x_bar = 0.5:")
print(df_fem.to_string())
print("\n" * 2)

# x_bar_2
# for every k ratio, calculate the heat convection from fdm and fem temp results
fdm_heat_results = []
fem_heat_results = []
for i, k_ratio in enumerate(data_dict["k_ratios"]):
    Bar.set_k_ratio(k_ratio, bar1, bar2) # Set the k value of bar2 = k*bar1
    dx = data_dict["section2"]["x_bar_2"]["x_values"][-1] -
        data_dict["section2"]["x_bar_2"]["x_values"][-2]
    (fdm_t0, fdm_t1) = data_dict["section2"]["x_bar_2"]["fdm_results"][i][-2:]
    fdm_heat_result = common.fdm_heat_extraction(fdm_t0, fdm_t1, dx, bar2, order=2)
    fdm_heat_results.append(fdm_heat_result)

    (fem_t0, fem_t1) = data_dict["section2"]["x_bar_2"]["fem_results"][i][-2:]
    fem_heat_result = common.fem_heat_extraction(fem_t0, fem_t1, dx, bar2)
    fem_heat_results.append(fem_heat_result)


data_dict["section2"]["x_bar_2"]["fdm_heat_results"] = np.array(fdm_heat_results)
data_dict["section2"]["x_bar_2"]["fem_heat_results"] = np.array(fem_heat_results)

df_fdm = pd.DataFrame(fdm_heat_results, index=[f'k2/k1 = {a}' for a in
    data_dict["k_ratios"]], columns=["Heat Convection"])
df_fem = pd.DataFrame(fem_heat_results, index=[f'k2/k1 = {a}' for a in
    data_dict["k_ratios"]], columns=["Heat Convection"])

print("FDM Heat Convection Results x_bar = 2/pi:")
print(df_fdm.to_string())
print("\n" * 2)

print("FEM Heat Convection Results x_bar = 2/pi:")
print(df_fem.to_string())
print("\n" * 2)

# Plotting x_bar_1 heat convection
plt.figure(figsize=(10, 6))
plt.plot(data_dict["k_ratios"], data_dict["section2"]["x_bar_1"]["fdm_heat_results"],
    label=f'FDM Heat Convection')
plt.plot(data_dict["k_ratios"], data_dict["section2"]["x_bar_1"]["fem_heat_results"],
    label=f'FEM Heat Convection')
#plt.plot(data_dict["k_ratios"], data_dict["section1"]["x_bar_1"]["heat_results"],
    label=f'Analytical Heat Convection')
```

```
    plt.xlabel('k2/k1')
    plt.ylabel('Q_Dot (W)')
    plt.legend()
    plt.grid(True)
    plt.savefig('images/section2/x_bar_1/heat_convection_vs_k_ratio.png', dpi=300)
    #plt.show()
    plt.clf()

    # Plotting x_bar_2 heat convection
    plt.figure(figsize=(10, 6))
    plt.plot(data_dict["k_ratios"], data_dict["section2"]["x_bar_2"]["fdm_heat_results"],
        label=f'FDM Heat Convection')
    plt.plot(data_dict["k_ratios"], data_dict["section2"]["x_bar_2"]["fem_heat_results"],
        label=f'FEM Heat Convection')
    #plt.plot(data_dict["k_ratios"], data_dict["section1"]["x_bar_2"]["heat_results"],
        label=f'Analytical Heat Convection')
    plt.xlabel('k2/k1')
    plt.ylabel('Q_Dot (W)')
    plt.legend()
    plt.grid(True)
    plt.savefig('images/section2/x_bar_2/heat_convection_vs_k_ratio.png', dpi=300)
    #plt.show()
    plt.clf()



def section_3():
    '''Convergence of FDM and FEM of interface temperature and heat convection'''
    print("Section 3: Convergence of FDM and FEM with Varying k Ratio")

    l = data_dict["length"]

    data_dict["section3"] = {}
    data_dict["section3"]["num_sections"] = [4, 8, 16, 32, 64, 128]

    for k, x_bar_1 in enumerate(data_dict["x_bar_values"]):
        # Calculate the number of rows needed for two columns of subplots
        num_k_ratios = len(data_dict["k_ratios"])
        num_rows = math.ceil(num_k_ratios / 2)

        # Define the figure for temperature convergence (two columns)
        fig_temp, axs_temp = plt.subplots(num_rows, 2, figsize=(15, 5 * num_rows))

        # Define the figure for heat convergence (two columns)
        fig_heat, axs_heat = plt.subplots(num_rows, 2, figsize=(15, 5 * num_rows))

        # Flatten the axs arrays for easier indexing
        axs_temp = axs_temp.flatten()
        axs_heat = axs_heat.flatten()
```

```
# For every k ratio, iterate through every number of points, calculate convergence
    (compared to analytical and extrapolated), print and add a subplot
for idx, k_ratio in enumerate(data_dict["k_ratios"]):
    Bar.set_k_ratio(k_ratio, bar1, bar2) # Set the k value of bar2 = k*bar1

    # Get the analytical data
    analy_interface_temp = case1.get_analytical_datapoints([x_bar_1], bar1, bar2,
        x_bar_1, l)
    analy_heat = case1.get_analytical_heat_convection(l, bar1, bar2, x_bar_1, l)

    # Setup data arrays
    fdm_interface_temps = []
    fem_interface_temps = []
    fdm_extrap_temps = [None, None]
    fem_extrap_temps = [None, None]
    fdm_heats = []
    fem_heats = []
    fdm_extrap_heats = [None, None]
    fem_extrap_heats = [None, None]
    fdm_interface_temp_errors = []
    fem_interface_temp_errors = []
    fdm_interface_temp_betas = [[None]]
    fem_interface_temp_betas = [[None]]
    fdm_heat_errors = []
    fem_heat_errors = []
    fdm_heat_betas = [[None]]
    fem_heat_betas = [[None]]
    fdm_interface_temp_extrap_errors = [None, None]
    fem_interface_temp_extrap_errors = [None, None]
    fdm_interface_temp_extrap_betas = [None, None]
    fem_interface_temp_extrap_betas = [None, None]
    fdm_heat_extrap_errors = [None, None]
    fem_heat_extrap_errors = [None, None]
    fdm_heat_extrap_betas = [None, None]
    fem_heat_extrap_betas = [None, None]

    # Get the fdm and fem data
    dx = 0
    for i, num_sections in enumerate(data_dict["section3"]["num_sections"]):
        fdm_temps, x_fdm =
            case1.solve(bar1=bar1,bar2=bar2,num_sections=num_sections,x_bar=x_bar_1,l=l,method="
        fem_temps, x_fem =
            case1.solve(bar1=bar1,bar2=bar2,num_sections=num_sections,x_bar=x_bar_1,l=l,method="
        # Get the interface temperature
        fdm_interface_temp = fdm_temps[num_sections // 2]
        fem_interface_temp = fem_temps[num_sections // 2]
        fdm_interface_temps.append(fdm_interface_temp)
        fem_interface_temps.append(fem_interface_temp)

        # Get the 2 last temperature to calculate heat convection
```

```
dx_prev = dx
dx = x_fdm[-1] - x_fdm[-2]
(fdm_t0, fdm_t1) = fdm_temps[-2:]
fdm_heat = common.fdm_heat_extraction(fdm_t0, fdm_t1, dx, bar2, order=2)
(fem_t0, fem_t1) = fem_temps[-2:]
fem_heat = common.fem_heat_extraction(fem_t0, fem_t1, dx, bar2)
fdm_heats.append(fdm_heat)
fem_heats.append(fem_heat)

# Calculated the Richardson extrpolation of interface temperature and heat
    convection
if i >= 2:
    fdm_extrap_temp = common.calc_extrapolated(fdm_interface_temps[-3],
        fdm_interface_temps[-2], fdm_interface_temps[-1])
    fem_extrap_temp = common.calc_extrapolated(fem_interface_temps[-3],
        fem_interface_temps[-2], fem_interface_temps[-1])
    fdm_extrap_heat = common.calc_extrapolated(fdm_heats[-3], fdm_heats[-2],
        fdm_heats[-1])
    fem_extrap_heat = common.calc_extrapolated(fem_heats[-3], fem_heats[-2],
        fem_heats[-1])
    fdm_extrap_temps.append(fdm_extrap_temp)
    fem_extrap_temps.append(fem_extrap_temp)
    fdm_extrap_heats.append(fdm_extrap_heat)
    fem_extrap_heats.append(fem_extrap_heat)

    # Calculate the errors in reference to extrapolated values
    fdm_interface_temp_extrap_error = common.calc_error(fdm_extrap_temp,
        fdm_interface_temp)
    fem_interface_temp_extrap_error = common.calc_error(fem_extrap_temp,
        fem_interface_temp)
    fdm_heat_extrap_error = common.calc_error(fdm_extrap_heat, fdm_heat)
    fem_heat_extrap_error = common.calc_error(fem_extrap_heat, fem_heat)
    fdm_interface_temp_extrap_errors.append(fdm_interface_temp_extrap_error)
    fem_interface_temp_extrap_errors.append(fem_interface_temp_extrap_error)
    fdm_heat_extrap_errors.append(fdm_heat_extrap_error)
    fem_heat_extrap_errors.append(fem_heat_extrap_error)

    # Calculate the betas in reference to extrapolated values
    fdm_interface_temp_extrap_beta = common.calc_beta(fdm_extrap_temp,
        fdm_interface_temps[-1], fdm_interface_temps[-2], dx, dx_prev)
    fem_interface_temp_extrap_beta = common.calc_beta(fem_extrap_temp,
        fem_interface_temps[-1], fem_interface_temps[-2], dx, dx_prev)
    fdm_heat_extrap_beta = common.calc_beta(fdm_extrap_heat, fdm_heats[-1],
        fdm_heats[-2], dx, dx_prev)
    fem_heat_extrap_beta = common.calc_beta(fem_extrap_heat, fem_heats[-1],
        fem_heats[-2], dx, dx_prev)
    fdm_interface_temp_extrap_betas.append(fdm_interface_temp_extrap_beta)
    fem_interface_temp_extrap_betas.append(fem_interface_temp_extrap_beta)
    fdm_heat_extrap_betas.append(fdm_heat_extrap_beta)
    fem_heat_extrap_betas.append(fem_heat_extrap_beta)
```

```
    # Calculate the error and convergence rates for fdm temp, fem temp, fdm
        heat, and fem heat
    fdm_interface_temp_error = common.calc_error(analy_interface_temp,
        fdm_interface_temp)
    fem_interface_temp_error = common.calc_error(analy_interface_temp,
        fem_interface_temp)
    fdm_heat_error = common.calc_error(analy_heat, fdm_heat)
    fem_heat_error = common.calc_error(analy_heat, fem_heat)
    fdm_interface_temp_errors.append(fdm_interface_temp_error)
    fem_interface_temp_errors.append(fem_interface_temp_error)
    fdm_heat_errors.append(fdm_heat_error)
    fem_heat_errors.append(fem_heat_error)

    if i >= 1: # if i == 0 then we cannot calculate convergence
        fdm_interface_temp_beta = common.calc_beta(analy_interface_temp,
            fdm_interface_temps[-1], fdm_interface_temps[-2], dx, dx_prev)
        fem_interface_temp_beta = common.calc_beta(analy_interface_temp,
            fem_interface_temps[-1], fem_interface_temps[-2], dx, dx_prev)
        fdm_heat_beta = common.calc_beta(analy_heat, fdm_heats[-1],
            fdm_heats[-2], dx, dx_prev)
        fem_heat_beta = common.calc_beta(analy_heat, fem_heats[-1],
            fem_heats[-2], dx, dx_prev)
        fdm_interface_temp_betas.append(fdm_interface_temp_beta)
        fem_interface_temp_betas.append(fem_interface_temp_beta)
        fdm_heat_betas.append(fdm_heat_beta)
        fem_heat_betas.append(fem_heat_beta)


# Print the interface temps for this k ratio
table_data = []
for i in range(len(data_dict["section3"]["num_sections"])):
    table_data.append([
        analy_interface_temp[0], # True T

        fdm_interface_temps[i], # fdm result
        fdm_extrap_temps[i],
        fdm_interface_temp_errors[i][0], # fdm % error in reference to analytical
        fdm_interface_temp_betas[i][0], # fdm beta
        fdm_interface_temp_extrap_errors[i], # fdm % error in reference to
            extrapolated value
        fdm_interface_temp_extrap_betas[i], # fdm beta

        fem_interface_temps[i], # fem result
        fem_extrap_temps[i],
        fem_interface_temp_errors[i][0], # fem % error
        fem_interface_temp_betas[i][0], # fem beta
        fem_interface_temp_extrap_errors[i], # fem extrap % error
        fem_interface_temp_extrap_betas[i], # fem beta
```

```
    ])

columns = ['True T', 'FDM T', 'FDM Extrap T', 'FDM Exact % Error', 'FDM Exact
    B', 'FDM Extrap % Error', 'FDM Extrap B', 'FEM T', 'FEM Extrap T', 'FEM
    Exact % Error', 'FEM Exact B', 'FEM Extrap % Error', 'FEM Extrap B']
df = pd.DataFrame(table_data, index=[f'N = {i}' for i in
    data_dict["section3"]["num_sections"]], columns=columns)

print(f"Convergence of FDM and FEM Temperature at x_bar = {x_bar_1:.3f} and
    k_ratio = {k_ratio}")
print(df.to_string())
print("\n" * 2)

# Print the heat convection results for this k ratio
table_data = []
for i in range(len(data_dict["section3"]["num_sections"])):
    table_data.append([
        analy_heat, # True Q
        fdm_heats[i], # fdm result
        fdm_extrap_heats[i], # fdm extrapolated heat
        fdm_heat_errors[i], # fdm % error in reference to analytical
        fdm_heat_betas[i], # fdm beta in reference to analytical
        fdm_heat_extrap_errors[i], # fdm % error in reference to extrapolated
            value
        fdm_heat_extrap_betas[i], # fdm beta

        fem_heats[i], # fem result
        fem_extrap_heats[i], # fem extrapolated heat
        fem_heat_errors[i], # fem % error
        fem_heat_betas[i], # fem beta
        fem_heat_extrap_errors[i], # fem % error
        fem_heat_extrap_betas[i] # fem beta
    ])

columns = ['True Q', 'FDM Q', 'FDM Extrap Q', 'FDM Exact % Error', 'FDM Exact
    B', 'FDM Extrap % Error', 'FDM Extrap B', 'FEM Q', 'FEM Extrap Q', 'FEM
    Exact % Error', 'FEM Exact B', 'FEM Extrap % Error', 'FEM Extrap B']
df = pd.DataFrame(table_data, index=[f'N = {i}' for i in
    data_dict["section3"]["num_sections"]], columns=columns)

print(f"Convergence of FDM and FEM Heat Convection with x_bar = {x_bar_1:.3f}
    and k_ratio = {k_ratio}")
print(df.to_string())
print("\n" * 2)

# Add a subplot of the interface temp convergence for this k ratio
ax_temp = axs_temp[idx]

# Data to plot for temperature convergence
num_sections = data_dict["section3"]["num_sections"]
```

```
    fdm_exact_errors = [e[0] for e in fdm_interface_temp_errors]
    fem_exact_errors = [e[0] for e in fem_interface_temp_errors]
    fdm_extrap_errors = fdm_interface_temp_extrap_errors
    fem_extrap_errors = fem_interface_temp_extrap_errors

    # Plotting temperature lines
    ax_temp.plot(num_sections, fdm_exact_errors, label="FDM Exact")
    ax_temp.plot(num_sections, fem_exact_errors, label="FEM Exact")
    ax_temp.plot(num_sections, fdm_extrap_errors, label="FDM Extrap")
    ax_temp.plot(num_sections, fem_extrap_errors, label="FEM Extrap")

    ax_temp.set_xscale("log")
    ax_temp.set_yscale("log")
    ax_temp.set_xlabel("Number of Sections (N)")
    ax_temp.set_ylabel("% Error")
    ax_temp.set_title(f"k_ratio = {k_ratio}")
    ax_temp.grid(True)
    ax_temp.legend()

    # Add a subplot of the heat convergence for this k ratio
    ax_heat = axs_heat[idx]

    # Data to plot for heat convergence
    fdm_exact_heat_errors = fdm_heat_errors
    fem_exact_heat_errors = fem_heat_errors
    fdm_extrap_heat_errors = fdm_heat_extrap_errors
    fem_extrap_heat_errors = fem_heat_extrap_errors

    # Plotting heat lines
    ax_heat.plot(num_sections, fdm_exact_heat_errors, label="FDM Exact")
    ax_heat.plot(num_sections, fem_exact_heat_errors, label="FEM Exact")
    ax_heat.plot(num_sections, fdm_extrap_heat_errors, label="FDM Extrap")
    ax_heat.plot(num_sections, fem_extrap_heat_errors, label="FEM Extrap")

    ax_heat.set_xscale("log")
    ax_heat.set_yscale("log")
    ax_heat.set_xlabel("Number of Sections (N)")
    ax_heat.set_ylabel("% Error")
    ax_heat.set_title(f"k_ratio = {k_ratio}")
    ax_heat.grid(True)
    ax_heat.legend()

# Hide any unused subplots (in case of an odd number of k_ratios)
for i in range(num_k_ratios, len(axs_temp)):
    fig_temp.delaxes(axs_temp[i])
    fig_heat.delaxes(axs_heat[i])

# Adjust layout for better spacing
fig_temp.tight_layout(rect=[0, 0.03, 1, 0.95])
fig_heat.tight_layout(rect=[0, 0.03, 1, 0.95])
```

```
        # Save the plots
        if k == 0:
            fig_temp.savefig("images/section3/x_bar_1/temp_convergences.png", dpi=300)
            fig_heat.savefig("images/section3/x_bar_1/heat_convergences.png", dpi=300)
        else:
            fig_temp.savefig("images/section3/x_bar_2/temp_convergences.png", dpi=300)
            fig_heat.savefig("images/section3/x_bar_2/heat_convergences.png", dpi=300)



def main():
    # Make directories for plots
    import os
    import shutil

    base_dir = "images"
    sub_dirs = ["section1", "section2", "section3"]
    nested_dirs = ["x_bar_1", "x_bar_2"]

    # Create the base directory if it doesn't exist
    if not os.path.exists(base_dir):
        os.mkdir(base_dir)

    # Create or clear subdirectories and their nested directories
    for sub_dir in sub_dirs:
        section_path = os.path.join(base_dir, sub_dir)
        if os.path.exists(section_path):
            # Remove all contents of the directory
            shutil.rmtree(section_path)
        # Create the section directory
        os.makedirs(section_path)

        # Create nested directories within each section
        for nested_dir in nested_dirs:
            nested_path = os.path.join(section_path, nested_dir)
            os.makedirs(nested_path)

    # import sys

    # # Redirect all print statements to a file
    # sys.stdout = open("output.txt", "w", encoding="utf-8")

    section_1() # Analytical solutions
    section_2() # FDM and FEM temperature and heat convection
    section_3() # Convergence of FDM and FEM temperature and heat convection

    # # Restore original stdout and close the file
    # sys.stdout.close()
    # sys.stdout = sys.__stdout__
```

```
if __name__ == "__main__":
    main()
```

5.2 case1.py

```
# case1.py

import numpy as np
import matplotlib.pyplot as plt

from Bar import Bar, set_k_ratio

def get_analytical_constants(bar1:'Bar', bar2:'Bar', x_bar=2/np.pi, l=1):
    '''Solve for C1, D1, C2, and D2
    Returns np.array([C1, D1, C2, D2])'''
    epsilon = (bar1.k*bar1.area*bar1.alpha) / (bar2.k*bar2.area*bar2.alpha)
    A = np.array([
        #C1                                  D1            C2                              D2
        [0,                                  1,            0,                              0],
        [0,                                  0,            np.sinh(bar2.alpha*l),
            np.cosh(bar2.alpha*l)],
        [np.sinh(bar1.alpha*x_bar),           0,           -1*np.sinh(bar2.alpha*x_bar),
            -1*np.cosh(bar2.alpha*x_bar)],
        [epsilon*np.cosh(bar1.alpha*x_bar),  0,           -1*np.cosh(bar2.alpha*x_bar),
            -1*np.sinh(bar2.alpha*x_bar)]
    ])

    B = np.array([
        0,
        100,
        0,
        0
    ])

    return np.linalg.solve(A, B)

def get_analytical_datapoints(x_points, bar1:'Bar', bar2:'Bar', x_bar=2/np.pi, l=1):
    '''Generates the temperature distribution given x_points.
    Returns np.array of tempature values'''
    constants = get_analytical_constants(bar1=bar1,bar2=bar2,x_bar=x_bar,l=1)
    C1, D1, C2, D2 = constants[0], constants[1], constants[2], constants[3]

    # Two temp functions for the left and right side of the interface
    left_temp = lambda x: C1*np.sinh(bar1.alpha*x) + D1*np.cosh(bar1.alpha*x)
    right_temp = lambda x: C2*np.sinh(bar2.alpha*x) + D2*np.cosh(bar2.alpha*x)
```

```
    temp_values = np.array([])
    for x in x_points:
        if x <= x_bar:
            temp = left_temp(x)
        else:
            temp = right_temp(x)
        temp_values = np.append(temp_values, temp)

    return temp_values

def get_analytical_heat_convection(x, bar1:'Bar', bar2:'Bar', x_bar=2/np.pi, l=1):
    '''Gets the analytical heat convection at x from heat flux conservation'''
    constants = get_analytical_constants(bar1=bar1,bar2=bar2,x_bar=x_bar,l=1)
    C1, D1, C2, D2 = constants[0], constants[1], constants[2], constants[3]

    t_prime = C2*bar2.alpha*np.cosh(bar2.alpha*x) + D2*bar2.alpha*np.sinh(bar2.alpha*x)

    q_dot = -bar2.k*bar2.area*t_prime
    return q_dot

def fdm_array(bar1:'Bar', bar2:'Bar', num_sections: int=6, x_bar=2/np.pi, l=1):
    '''num_sections = total sections to cut the bar. Note that this is NOT evenly
        distributed over the entire
    bar. Half of the sections are left of x_bar, half are to the right.
    Returns the A matrix of the FDM equation Ax = B'''
    dx1 = x_bar / (num_sections / 2)
    dx2 = (l - x_bar) / (num_sections / 2)
    k1 = 2 + bar1.alpha**2 * dx1**2
    k2 = 2 + bar2.alpha**2 * dx2**2

    beta1 = (bar1.k * bar1.area) / (dx1)
    beta2 = (bar2.k * bar2.area) / (dx2)

    #rho = beta1 - beta2 - (bar1.k * bar1.area * dx1 * bar1.alpha**2) / 2 + (bar2.k *
        bar2.area * dx2 * bar2.alpha**2) / 2
    rho = beta1 + beta2 + (bar1.k * bar1.area * dx1 * bar1.alpha**2) / 2 + (bar2.k *
        bar2.area * dx2 * bar2.alpha**2) / 2
    # Making the matrix
    # There are num_sections - 1 rows and columns
    # row 0, row n - 2, and row num_sections/2 - 1 are unique
    # the rest are -1, k, -1 where k changes from k1 to k2 at num_sections/2

    # Initialize matrix A with zeros
    A = np.zeros((num_sections - 1, num_sections - 1))

    # Fill the matrix A
    for row in range(num_sections - 1):
        if row == 0:
            # First row for boundary at x = 0 (for example Dirichlet or Neumann condition)
            A[row, row] = k1
```

```
            A[row, row+1] = -1
        elif row == num_sections // 2 - 1:
            # Special row at the interface between bar1 and bar2
            A[row, row-1] = -beta1
            A[row, row] = rho # This is the rho value you computed
            A[row, row+1] = -beta2
        elif row == num_sections - 2:
            # Last row for boundary at x = l (again assuming Dirichlet or Neumann condition)
            A[row, row-1] = -1
            A[row, row] = k2
        else:
            # Interior rows for bar1 and bar2 (uniform grid)
            if row < num_sections // 2 - 1:
                # In bar1 region
                A[row, row-1] = -1
                A[row, row] = k1
                A[row, row+1] = -1
            else:
                # In bar2 region
                A[row, row-1] = -1
                A[row, row] = k2
                A[row, row+1] = -1
    return A

def fem_array(bar1:'Bar', bar2:'Bar', num_sections: int=6, x_bar=2/np.pi, l=1):
    '''num_sections = total sections to cut the bar. Note that this is NOT evenly
        distributed over the entire
    bar. Half of the sections are left of x_bar, half are to the right.
    Returns the A matrix of the FEM equation Ax = B'''
    dx1 = x_bar / (num_sections / 2)
    dx2 = (l - x_bar) / (num_sections / 2)
    k1 = (2/dx1 + 4*bar1.alpha**2*dx1/6) / (1/dx1 - bar1.alpha**2*dx1/6)
    k2 = (2/dx2 + 4*bar2.alpha**2*dx2/6) / (1/dx2 - bar2.alpha**2*dx2/6)

    beta1 = (bar1.k * bar1.area) / (dx1)
    beta2 = (bar2.k * bar2.area) / (dx2)

    rho = beta1 + beta2 + (bar1.k * bar1.area * dx1 * bar1.alpha**2) / 2 + (bar2.k *
        bar2.area * dx2 * bar2.alpha**2) / 2
    # Making the matrix
    # There are num_sections - 1 rows and columns
    # row 0, row n - 2, and row num_sections/2 - 1 are unique
    # the rest are -1, k, -1 where k changes from k1 to k2 at num_sections/2

    # Initialize matrix A with zeros
    A = np.zeros((num_sections - 1, num_sections - 1))

    # Fill the matrix A
    for row in range(num_sections - 1):
        if row == 0:
```

```
            # First row for boundary at x = 0 (for example Dirichlet or Neumann condition)
            A[row, row] = k1
            A[row, row+1] = -1
        elif row == num_sections // 2 - 1:
            # Special row at the interface between bar1 and bar2
            A[row, row-1] = -beta1
            A[row, row] = rho # This is the rho value you computed
            A[row, row+1] = -beta2
        elif row == num_sections - 2:
            # Last row for boundary at x = l (again assuming Dirichlet or Neumann condition)
            A[row, row-1] = -1
            A[row, row] = k2
        else:
            # Interior rows for bar1 and bar2 (uniform grid)
            if row < num_sections // 2 - 1:
                # In bar1 region
                A[row, row-1] = -1
                A[row, row] = k1
                A[row, row+1] = -1
            else:
                # In bar2 region
                A[row, row-1] = -1
                A[row, row] = k2
                A[row, row+1] = -1
    return A

def solve(bar1:'Bar', bar2:'Bar', num_sections: int=6, x_bar=2/np.pi, l=1, method="FDM"):
    '''Solves using FDM or FEM as specified. Returns (temps, x_values)'''
    if method == "FDM":
        A = fdm_array(bar1,
                      bar2,
                      num_sections=num_sections,
                      x_bar=x_bar,
                      l=l)
    elif method == "FEM":
        A = fem_array(bar1,
                      bar2,
                      num_sections=num_sections,
                      x_bar=x_bar,
                      l=l)

    B = np.zeros((num_sections - 1))
    B[-1] = 100

    # Add boundary conditions
    interior_temps = np.linalg.solve(A, B)
    all_temps = np.array([0])
    all_temps = np.append(all_temps, interior_temps)
    all_temps = np.append(all_temps, 100)
```

```
    # Generate the x_values to return along with the temps
    dx2 = (l - x_bar) / (num_sections / 2)
    x_values = np.linspace(0, x_bar, num_sections // 2 + 1) # [0, x_bar]
    x_values = np.append(x_values, np.linspace(x_bar+dx2, l, num_sections // 2))
    return all_temps, x_values


if __name__ == "__main__":
    bar1 = Bar(
        radius=0.1,
        k=0.5,
        h=0.25
    )
    bar2 = Bar(
        radius=0.1,
        k=2,
        h=0.25
    )

    set_k_ratio(0.5, bar1, bar2)

    x_bar = 0.5

    x_values = np.linspace(0, 1, 1000)
    temp_values = get_analytical_datapoints(x_points=x_values,
                                            bar1=bar1,
                                            bar2=bar2,
                                            x_bar=x_bar)


    # fdm tests
    l = 1
    num_sections = 4
    dx1 = x_bar / (num_sections / 2)
    dx2 = (l - x_bar) / (num_sections / 2)
    fdm_temps, x_fdm = solve(bar1=bar1,
                   bar2=bar2,
                   num_sections=num_sections,
                   x_bar=x_bar,
                   l=1,
                   method="FDM")
    fem_temps, x_fem = solve(bar1=bar1,
                   bar2=bar2,
                   num_sections=num_sections,
                   x_bar=x_bar,
                   l=1,
                   method="FEM")

    # Plot the data
    plt.plot(x_values, temp_values, label='Temperature Distribution')
```

```
    plt.axvline(x=x_bar, color='r', linestyle='--', label='x_bar')
    plt.plot(x_fdm, fdm_temps, 'o-', label='Temperature Distribution (FDM)', color='g')
    plt.plot(x_fem, fem_temps, 'o-', label='Temperature Distribution (FEM)', color='r')
    plt.xlabel('x')
    plt.ylabel('Temperature')
    plt.title('Temperature Distribution Along the Bar')
    plt.legend()
    plt.grid(True)

    # Show plot
    plt.show()
```

5.3 Bar.py

```
# Bar.py

from numpy import pi

class Bar():
    def __init__(self, radius:float=0.1, k:float=0.5, h:float=0.0025):
        self.radius:float = radius
        self.k:float = k
        self.h:float = h

        self.update_properties()

    def update_properties(self):
        self.area:float = pi*self.radius**2
        self.p: float = 2*pi*self.radius
        self.alpha = ((self.h*self.p)/(self.k*self.area))**0.5

    def set_k(self, k:float):
        self.k = k
        self.update_properties()

def set_k_ratio(ratio: float, bar1:'Bar', bar2:'Bar'):
    '''Sets the value of bar2.k = ratio*bar1.k and updates internal properties'''
    bar2.set_k(bar1.k*ratio)
```

5.4 common.py

```
# common.py

import numpy as np
from Bar import Bar

def fdm_heat_extraction(t_0, t_1, dx, bar:'Bar', order=2):
```

```
    '''Get the heat conduction at the point t_1 using Taylor series'''
    if order == 1:
        return -1 * bar.k * bar.area * (t_1 - t_0) / dx
    elif order == 2:
        return -1 * bar.k * bar.area * (((t_1 - t_0) / dx) + (bar.alpha**2 * dx * t_1 / 2))

def fem_heat_extraction(t_0, t_1, dx, bar:'Bar'):
    '''Get the heat conduction at the point t_1 using FEM equation'''
    term_1 = (-1/dx + bar.alpha**2*dx/6) * t_0
    term_2 = (1/dx + 2*bar.alpha**2*dx/6) * t_1
    return -1 * bar.k * bar.area * (term_1 + term_2)

def calc_error(exact, q_1):
    return np.abs((exact - q_1) / exact)

def calc_beta(exact, q_1, q_2, dx_1, dx_2):
    return np.log(np.abs((exact - q_1)/(exact - q_2))) / np.log(dx_1 / dx_2)

def calc_extrapolated(q1, q2, q3):
    '''Calcs the Richardson Extrapolation value based on 3 different meshes.
    Assumes that q3 is 2x finer than q2 is 2x finer than q1'''
    numerator = q1*q3 - q2**2
    denominator = q1 + q3 - 2*q2
    return numerator / denominator
```